On Approximation of Functions with some Conditions

Sukaina Al-Bairmani
College of Information Technology
PO box 4, Hilla, Iraq
Tel: +9647711650437 E-mail:sukeina_2009@yahoo.com

Abstract

The purpose of this paper is to obtain new results in approximation of functions by polynomials, we discuss if the function f satisfies S restrictions on its $0 \le t_i < t_{i+1}, i = 1, 2, \ldots, s$ where space which is $f \in L_P^{(2t_s)}$, $1 < p < \infty$, for some $t_s \ge 0$.

1. Introduction and main results:

Kimchi and Leviatan [Kimchi, E. & Leviatan, D., 1976], proved that restriction on its $0 \le k_1 < k_2 < \ldots < k_r \le n$ where $f \in C^{(2k_s)}[-1, 1]$,for some fixed integers $k_r \ge 0$, be such that $\sum_n \left(1/n\right) \omega\left(f^{(2k_r)}, 1/\sqrt{n}\right) < \infty$, in my paper we used another space which is $L_P^{(2t_s)}$, $1 < p < \infty$,we prove that the function f and the best polynomial approximation to f satisfies same restrictions for the sufficiently large n,

We must take three functions, $f \in L_P^{(2t_s)}, 1 < p < \infty, u_i(x), v_i(x), i = 1, 2, \ldots, s, \ x \in [-1, 1]$ satisfies

$$u_i(x) < f^{(t_i)}(x) < v_i(x), \quad i = 1, 2, ..., s, x \in [-1, 1]$$

And may be $u_i(x) = -\infty, v_i(x) = \infty$, where $0 \le t_1 < t_2 < \ldots < t_s \le n$ be fixed integer and let $p_n \in \pi_n$ such that

$$u_i(x) < p^{(t_i)}(x) < v_i(x), \quad i = 1, 2, ..., s, x \in [-1, 1]$$

When π_n is the class of all polynomials of degree $\le n$.

Our result is therefore

Theorem: There are constants $M_q, q = 1, 2, \ldots, t$ with the following property: If ω is a modulus of smoothness for which

$$\sum_n \left(1/n\right) \omega\left(f^{(2t_s)}, 1/\sqrt{n}\right)_p < \infty$$

And if $f \in L_P^{(2t_s)}, 1 < p < \infty$ for some $t_s \ge 0$, and 2S fixed functions $u_i(x), v_i(x), i = 1, 2, \ldots, s, \ x \in [-1, 1]$, then for the sufficiently large n there exist $p_n \in \pi_n$ the best polynomial approximation of f satisfies

$$u_i(x) < p^{(t_i)}(x) < v_i(x), \quad i = 1, 2, ..., s, x \in [-1, 1]$$

Such that $\left\| f^{(q)}(x) - p_n^{(q)}(x) \right\|_p \le M_q \sum_{j \ge n} \frac{1}{k} \phi\left(\frac{1}{k}\right)_p$, $q = 1, 2, \ldots, t_s$

For some fixed integer $0 \le t_i < t_{i+1}, i = 1, 2, \ldots, s$.

2. Proofs: to prove this theorem we need the following lemma

Lemma: let $f \in L_P^{(2t_s)}$, $1 < p < \infty$ and $p_n(x)$ of degree $n = 1, 2, \ldots,$

Where $\left\| f(x) - p_n(x) \right\|_p \le \left(\Delta_n(x)\right)^q \phi\left(\Delta_n(x)\right)_p$, $x \in [-1, 1]$.

Then f has continuous derivatives $f', \ldots, f^{(q)}$ where ϕ is the modulus of smoothness for which

$$\sum_{n=1}^{\infty} \frac{1}{n} \phi\left(\frac{1}{n}\right)_p < \infty ,$$

Such that

$$\left\| f^{(q)}(x) - p_n^{(q)}(x) \right\|_p \le M_q \sum_{n \ge (\Delta_n(x))^{-1}} \frac{1}{n} \phi\left(\frac{1}{n}\right)_p, \quad q = 1, 2, \ldots, t_s. \quad \textbf{(1)}$$

For each $q = 1, 2, \ldots, t$ there is M_q constants depends upon q .

2. 1. Proof of Lemma:

Let $$f(x) = p_n + \sum_{i=1}^{\infty} \left(p_{2^i n}(x) - p_{2^{i-1} n}(x) \right)$$

Since $p_n(x)$ is the polynomial of best approximation of degree n for f then

$$f(x) - p_n(x) + \sum_{i=1}^{\infty} \left(p_{2^i n}(x) - p_{2^{i-1} n}(x) \right)$$

This series converges uniformly. This series obtained by format differentiation,

$$p_n^{(q)}(x) + \sum_{i=1}^{\infty} \left(p_{2^i n}^{(q)}(x) - p_{2^{i-1} n}^{(q)}(x) \right)$$

Also converge uniformly as a converges as a consequence and thus represents

$f^{(q)} - p_n^{(q)}$ For

$$\left\| p_{2^i n}(x) - p_{2^{i-1} n}(x) \right\|_p \le 2 \left(\Delta_{2^{i-1} n}(x)\right)^q \phi\left(\Delta_{2^{i-1} n}(x)\right)_p$$

Now, we note that

$$\frac{1}{4}\Delta_n(x) \le \Delta_{2n}(x) \le \frac{1}{2}\Delta_n(x),\ x \in [-1,\ 1],\ n = 1, 2, \ldots$$

Get the estimate

$$\left\| p^{(q)}_{2^i n}(x) - p^{(q)}_{2^{i-1} n}(x) \right\|_p = O(1)\ \phi\left(\Delta_{2^i n}(x)\right)_p$$

And $f^{(q)} - p^{(q)}_n$

Is equal to $O(1)\left(1\sum_{i=1}^{\infty}\phi\left(\Delta_{2^i n}(x)\right)_p\right) = O(1) \sum_{\frac{\Delta 2n(x)^{-1}}{2}} \frac{1}{n}\ \phi\left(\frac{1}{n}\right)_p$

Then the last sum tends to zero uniformly in x.

Since $\frac{1}{2}\Delta_{2n}(x)^{-1} \ge \Delta_n(x)^{-1} \ge n$, [Lorentz, G.G., 1966] **(2)**

Where $\Delta_n(x) = \max\left\{ \sqrt{1-x^2}/n\ ,\ 1/n^2 \right\},\ x \in [-1, 1],\ n - 1, 2, \ldots, \Delta_0(x) = 1$

2. 2. Proof of Theorem: $f \in L_P^{(2t_s)}, 1 < p < \infty,\ p_n \in \pi_n$ is the best Polynomial approximation to f,

And $E_n(f) = \inf_{p \in \pi_n} \|f - p\|_p = \|f - p\|_p \le \frac{H}{n^{2t_s}}\, \omega\left(\frac{1}{n}\right)_p$, by Jackson's theorems [AL-Bermani, S. A., 2008]

Where ω is the modulus of smoothness of f,

Then $\left\| f(x) - p_n(x) \right\|_p \le \frac{H}{n^{2q}}\ \omega\left(\frac{1}{n}\right)_p \le H\left(\Delta_n(x)\right)^q \omega\left(\sqrt{\Delta_n(x)}\right)_p, q = 1, 2, \ldots, t_s$

For which $\frac{1}{n^2} \le \Delta_n(x) \le \frac{1}{n},\ x \in [-1,\ 1]$

Assume that $\phi(f) = H\,\omega\left(\sqrt{f}\right)$

We get $\left(\frac{1}{f}\right)\phi\left(\frac{1}{n}\right) = \left(\frac{H}{n}\right)\omega\left(1/\sqrt{n}\right),$ **(3)**

And $\left\| f(x) - p_n(x) \right\|_p \leq_p (\Delta_n(x))^q \phi\left(\sqrt{\Delta_n(x)}\right)_p, q = 1, 2, \ldots, t_s$

Hence by (3) so that

$$\sum_n \left(1/n\right) \omega\left(f^{(2t_s)}, 1/\sqrt{n} \right)_p < \infty \quad , \quad \text{Implies } \sum_n \left(1/n\right) \phi\left(1/n\right)_p < \infty$$

Then $p_n \in \pi_n$ the best polynomial approximation of f satisfies (1),

Thus For each $q = t_{a,} a = 1, 2, \ldots, s,$

$$\left\| f^{(q)}(x) - p_n^{(q)}(x) \right\|_p \leq M_q \sum_{n \geq (\Delta_n(x))^{-1}} \frac{1}{k} \phi\left(\frac{1}{k}\right)_p$$

$$\leq M_q \sum_{j \geq n} \frac{1}{k} \phi\left(\frac{1}{k}\right)_p, \quad q = 1, 2, \ldots, t_s. \qquad \textbf{(4)}$$

Where we applied (2)

Really the sum on the right-hand side of (4) tends to 0 as $n \to \infty$, uniformly in $-1 \leq x \leq 1$.
Thus the best polynomial approximation is also the best restricted derivatives approximation.

References

Kimchi, E. and Leviatan, D., (1976), On Restricted Best Approximation to Functions with Restricted Derivatives, SIAM J. Math, Vol. 13 No. , PP. 51-53.

Lorentz, G.G., (1966), Approximation of Function, Holt, Rinehart and Winston, New York.

AL-Bermani, S. A., (2008), Estimations for monotone approximation
, M.Sc. In Mathematics, College of Education, Babylon University.