Oscillation Criteria for Second Order Nonlinear Differential Equations

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Abstract
We present new oscillation criteria for certain nonlinear differential equations of second order with damping term

$$(r(t)x'(t))'+p(t)x'(t)+q(t)f(x(t))=0 \qquad t\geq t_0 \qquad (1)$$

to used the oscillatory solutions of differential equatioins

$$(\alpha(t)x'(t))'+\beta(t)f(x(t))=0 \qquad (2)$$

where that are different from most known ones. Our results extend and improve some previous oscillation criteria and cover the cases which are not covered by known results. In this paper, by using the generalized Riccati technique we get a new oscillation and nonoscillation criteria for (1). The theorems prove to be efficient in many cases and give the results in the literature.

Key words: differential, equations, interval, criteria, damping, second order etc.

Introduction

In this paper we are being consider the oscillation solutions in the second order nonlinear functional differential equation :

$$(r(t)x'(t))'+q(t)x'(t)+p(t)f(x(t))=0, \quad t\geq t_0 \qquad (1)$$

where we very often use the following assumptions :

A1) $p(t)$ is a real valued and locally integrable function over $I=[\alpha,\infty)$ and not identically zero in any neighborhood of ∞.

A2) $q(t)$ is a real valued and locally integrable over I .

A3) For all $t\in I$, $r(t)>0$, for $t\in I=[\alpha,\infty)$, and $\int_\alpha^\infty \frac{1}{r(t)}dt=\infty$.

A4) $f\in C(R,R)$, $xf(x)>0$, and $f'(x)\geq k>0$.

By a solution of equation (1) or (2) we consider a function $x(t)$, $t\in[t_x,\infty)\subset[t_0,\infty\}$ which is twice continuously differentiable and satisfies equation (1) or (2) on the given interval. The number depends on that particular solution *x(t)* under consideration.

We consider only non-trivial solutions . A solution *x(t)* of (1) or (2) is said to be oscillatory if there exists a sequence $\{\lambda_n\}_{n=1}^\infty$ of points in the interval $[t_0,\infty\}$, so that $\lim_{n\to\infty}\lambda_n=\infty$ and $x(\lambda_n)=0$, $n\in N$, otherwise it is said to be nonoscillatory. An equation is said to be oscillatory if all its solutions are oscillatory, otherwise it is considered that is nonoscillatory solution .

An important tool in the study of oscillatory behavior of solutions of these equations is the averaging technique . We can se in this paper that the equation

$$(a(t)x'(t))' + p(t)f(x(t)) = 0 \qquad (2)$$

may be transform in the form (1) and the solution of (2) are also solution of (1).

Leighton's result (see [9]) where every solution of (2) when *f(x)=x* is oscillatory need to satisfy the conditions

$$\int_0^\infty \frac{1}{a(t)}dt = \infty$$

and

$$\int_0^\infty p(t) = \infty .$$

For the same case Willet in [10] obtain the following criteria : If

$$\int_0^\infty \frac{1}{a(t)}dt < \infty$$

and

$$\int_{t_0}^\infty a(t)\left(\int_t^\infty \frac{ds}{a(s)}\right)^2 dt = \infty$$

then every solution of (2) oscillates.

N. Yamaoka in [11] presented the result : If A_2 , A_3 hold and *a(t), b(t)* satisfy

$$a(t)b(t)\left(\int_t^\infty \frac{1}{a(l)}dl\right)^2 \geq 1$$

for *t* sufficiently small, and that there exists a $\lambda > \frac{1}{16}$, such that

$$\frac{f(x)}{x} \geq \frac{1}{4} + \frac{\lambda}{(\log|x|)^2}$$

for $|x|$ sufficiently small. Then all solutions of (2) are oscillatory.

Moreover equation include the differential equations where *f(x)=x* which recently had been discussed by R. Kim on [3].

On the continuation we present the well – known Gronwall's inequality.

Lemma 1. Let $I = [t_0, T)$, be an interval of real numbers , and suppose that

$$u(t) \leq c + \int_{t_0}^t q(s)u(s)ds , \text{ for } t \in I , \qquad (4)$$

where c is a nonnegative constant , and $u, q \in C(I, \mathfrak{R}^+)$. Then ,

$$u(t) \le c \exp\int_{t_0}^{t} q(s)ds, \text{ for } t \in I.$$

The Conditions for oscillatory solutions of the second order differential equations (1) are studied by many authors (see [2], [4], [6],[7] etc.) .Here , we give some conditions for coefficients where the equation (1) have oscillatory solutions and we also take in to account the result we have obtained in the previous researches , here we present more generalized criteria that define oscillation solution of the equation (1) to used oscillatory solutions of (2).

In this paper are presented theorems, that use generalized Riccati – type transformations, and averaging technique , which explain results for oscillatory nature of differential equations also, our results extend and improve a number of existing results (see [3], [8], etc.).

Main result

What follows is, $E(l)$, $\beta(l)$ denote

$$E(l) = e^{\int_{\alpha}^{t} \frac{q(l)}{r(l)}dl}$$

and

$$B(\alpha) = \int_{\alpha}^{\infty} (p(l) - \frac{q^2(l)}{4kr(l)})dl$$

that we will use in the following teorem.

Teorem 1 : The equation (1) is oscillatory if for $p(t) \ge 0$, $t \ge \alpha$, and

$$\int_{\alpha}^{\infty} \frac{1}{E(l)r(l)} dl = \infty \qquad (2)$$

$$B(\alpha) = \infty \qquad (3)$$

Proof: For $E(l) = e^{\int_{\alpha}^{t} \frac{q(l)}{r(l)}dl}$, we have $E'(t) = \frac{q(t)}{r(t)} E(t)$, where $W(t) > 0$, and the equation (1) is reduced in to

$$(E(t)r(t)x'(t))' + E(t)p(t)f(x(t)) = 0 \qquad (4)$$

We se that (1) is oscillatory if and only if the equation (4) is oscillatory.

Assume that (1) is nonoscillatory. Then there exists a nonoscillatory solution *x(t)* of (1) . So we may assume that $x(t) > 0$ on $[t_1, \infty)$, for some $t_1 > \alpha$. We show that $x'(t) > 0$, for $t \ge t_1$.

From (4) we obtain that

$$(E(t)r(t)x'(t))' = -E(t)p(t)x(t) \le 0$$

from what $E(t)r(t)x'(t)$ is not increasing for $t \ge t_1$. Assume that $E(t_2)r(t_2)x'(t_2) < 0$ for some $t_2 > t_1$. Put $E(t_2)r(t_2)x'(t_2) = L$, then for $t \ge t_2$, we have

$$E(t)r(t)x'(t) \le L.$$

Dividing both sides by $E(t)r(t)$ and integrating from t_2 to t $(> t_2)$, we obtain

$$x(t) - x(t_2) \le L\int_{t_2}^{t} \frac{1}{E(l)r(l)} dl$$

because $L\int_{t_2}^{t}\frac{1}{E(l)r(l)}dl$ is tending to $-\infty$, where $t\to\infty$, we conclude that $x(t)<0$, for sufficiently large t, which is a contradiction . Therefore $x'(t)>0$, for $t\geq t_1$.
In case that $x(t)<0$, put $y(t)=-x(t)$. So we have $x'(t)>0$.
Considering the function

$$W(t)=\frac{r(t)x(t)}{f(x(t))}$$

we have

$$W'(t)=-\frac{q(t)}{r(t)}W(t)-p(t)-\frac{W^2 f'(x(t))}{r(t)}$$

and from $f'(x)\geq k>0$, we obtain

$$W'(t)\leq-\frac{q(t)}{r(t)}W(t)-p(t)-\frac{W^2(t)k}{r(t)},$$

$$W'(t)\leq-\frac{1}{r(t)}(W(t)\sqrt{k}+\frac{q(t)}{2\sqrt{k}})^2+\frac{q^2(t)}{4kr(t)}-p(t) \qquad (5)$$

Integrating (5) from t_2 to t $(>t_2)$, we get

$$W(t)-W(t_1)+\int_{t_1}^{t}(p(l)-\frac{q^2(l)}{4kr(l)})dl\leq-\int_{t_1}^{t}\frac{1}{r(l)}(W(l)\sqrt{k}-\frac{q(l)}{2\sqrt{k}})^2\,dl\,.$$

By means of (4) there exists a $t_3\geq t_1$, such that for $t\geq t_3$, we gain

$$W(t)\leq-\int_{t_1}^{t}\frac{1}{r(l)}(W(l)\sqrt{k}-\frac{q(l)}{2\sqrt{k}})^2\,dl$$

which is impossible because $W(t)>0$, for $t\geq t_1$.

Lema 1 : Assume that for $t\geq\alpha$, $p(t)\geq 0$, $q(t)\geq 0$ and (3) are valid. If the differential equations (1) has a positive solution , we have

$$\lim_{t\to\infty}\frac{r(t)x'(t)}{f(x(t))}=0.$$

Proof : Let $x(t)>0$, be a solution of (1) . From Teorem 1. it follows that from $p(t)\geq 0$, for $t\geq\alpha$, and (3) that there exists a $t_1\geq\alpha$ such that $x'(t)>0$, for $t\geq t_1$.
Put

$$W(t)=\frac{r(t)x'(t)}{f(x(t))}>0$$

for $t\geq t_1$, and consider Ricati inequation

$$W'(t)\leq-\frac{q(t)}{r(t)}W(t)-p(t)-\frac{W^2(t)k}{r(t)}$$

it is obvious that

$$-\frac{W'(t)}{W(t)} \geq \frac{k}{r(t)}.$$

Integrating the above inequation over $[t_1,\infty)$ and considering the condition $r(t)>0$, and

$\int_{\alpha}^{\infty}\frac{1}{r(t)}dt=\infty$, we have $-\int_{\alpha}^{x}\frac{W'(t)}{W(t)}dt=\int_{\alpha}^{x}\frac{1}{r(t)}dt$

$$\frac{1}{W(x)}=\frac{1}{W(\alpha)}+\int_{\alpha}^{x}\frac{1}{r(t)}dt$$

from where for $x\to\infty$, we obtain

$$\lim_{x\to\infty}W(x)=0.$$

To used the result presented in [8] , here in follow we give the bounded solution of (2) with :

Theorem 2. Suppose $\alpha(t),\beta(t)\in C^1([t_0,\infty))$ and $(\alpha(t)\beta(t))'\geq 0$, for $t\geq t_0$ and

$$\int^{\pm\infty} f(l)dl=\infty, \qquad (6)$$

then the solution $x(t)$ of the equation (2) such that $x(t_1)=0,\ t_1\geq t_0$ for some $t_1\in[t_0,\infty)$ is bounded.

Proof : Let $x(t)$ be a arbitrary solution of equation (1) such that $x(t_1)=0,\ t_1\geq t_0$.

Put

$$F(s)=\int_{x(t_1)}^{s} f(s)dl. \qquad (7)$$

Multiplying the equation (2) by $\alpha(t)x'(t)$, we gain

$$\frac{1}{2}((\alpha(t)x'(t))^2)'+\alpha(t)x'(t)\beta(t)f(x(t))=0$$

integrating from t_1 to t, we obtain

$$\frac{1}{2}(\alpha(t)x'(t))^2-\frac{1}{2}(\alpha(t_1)x'(t_1))^2+$$

$$+\alpha(t)\beta(t)F(x(t))-\int_{t_1}^{t}(\alpha(s)\beta(s))'F(x(s))ds=0. \qquad (8)$$

Denote

$$c=\frac{1}{2}(\alpha(t_1)x'(t_1))^2$$

now by (8) it follows that :

$$\alpha(t)\beta(t)F(x(t)\leq c+\int_{t_1}^{t}(\alpha(s)\beta(s))'F(x(s))ds=c+\int_{t_1}^{t}\frac{(\alpha(s)\beta(s))'}{\alpha(s)\beta(s)}\alpha(s)\beta(s)F(x(s))ds. \qquad (9)$$

Hence , by Granwalls-it inequality , we get

$$\alpha(t)\beta(t)F(x(t)\leq c\cdot e^{\int_{t_1}^{t}\frac{(\alpha(s)\beta(s))'}{\alpha(s)\beta(s)}ds} \qquad (10)$$

from this

$$F(x(t)) \leq \frac{c}{\alpha(t_1)\beta(t_1)}, \quad t_1 > t_0 > 0 \tag{11}$$

so, $F(x(t))$ is bounded and from (6) the solution $x(t)$ is bounded .

Example: Consider differential equation

$$(\frac{1}{e^t}x'(t))' + \frac{1}{t^2 \ln t}x'(t) + \frac{1}{1+e^t}(x(t)+x^3(t)) = 0, \quad t > 0 \tag{12}$$

for $r(t) = \frac{1}{e^t}$, $q(t) = \frac{2}{1+e^t}$ and $f(x) = x + x^3$,

from where $f'(x) = 1 + 3x^2 \geq 1 = k > 0$.

Also , for

$$E(l) = e^{\int_\alpha^l \frac{e^t}{1+e^t}dt} = e^t - e^\alpha, \text{ we have } \quad \int_\alpha^\infty \frac{1}{E(l)r(l)}dl = \infty$$

and

$$B(\alpha) = \int_\alpha^\infty (p(l) - \frac{q^2(l)}{4kr(l)})dl = \int_\alpha^\infty (\frac{1}{t\ln t} - \frac{e^t}{4k(1+e^t)^2})dt = \infty$$

that from the theorem 1. the equation (12) is oscillatory.

For instance, if the mass may have to encounter air resistance in it's motion. A typical physical assumption is that the frictional force points opposite to the direction of motion and has magnitude proportional to the speed. If there is no external forces, this leads to the differential equation of the form

$$mu''(t) = \gamma(t)u(t)' + k(t)u(t) \tag{13}$$

(where $\gamma u'$ is magnitude of frictional force)

From Hook's Law, which claims that if the string is stretched (or compressed) x units from its natural length

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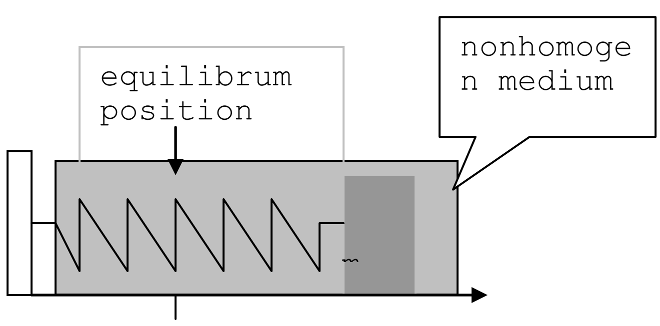

fig.

it exerts a force that is proportional to x :

$$\text{restoring force} = -\gamma x - ku$$

where k is a positive constant (called the spring constant) . If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law

(force equals mass time acceleration), we have (13) .
One of the choices, where for $m, k, \gamma = positive\ const$ we have the characteristic equation

$$mr^2 + \gamma r + k = 0$$

with roots

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}.$$

From the last equation we have three cases for the general solution, but oscillation solution we have for $\gamma^2 < 4km$ and the general solution has the form

$$u(t) = Ae^{r_1 t} + Be^{r_2 t}.$$

For $\alpha = \sqrt{4km - \gamma^2}$, we have the general solution and it has the following form

$$u(t) = e^{-\frac{\gamma}{2m}t}(A\cos(\alpha t) + B\sin(\alpha t)) = \mathrm{Re}^{-\frac{\gamma}{2m}t}\cos(\alpha t - \delta)$$

where $R = \sqrt{A^2 + B^2}$, and $\delta = \tan(\frac{B}{A})$.

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Short CV

Xhevair Beqiri was born in 20.10.1965 in Koroshishta municipalyti of Struga Macedonia. I have graduated and holding a degree from University of Prishtina , Kosova and postgraduated from the same University; doctorate studies in University of Tirana , Albania of department of mathematics, since .