Asymptotic Approximation of the Second Order Partial Differential Equation by Using Many Functions

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Abstract

We obtain an asymptotic expansion, containing regular boundary corner functions in the small parameter ε, for the solution of a second order partial differential equation. We construct the asymptotic expansion $u_n(x,t,\varepsilon)$ for the modified problem and prove it is the unique solution. Also we have proved that the solution is valid uniformly in the domain ϖ, and the asymptotic approximation is within $O(\varepsilon^{n+1})$.

1- Introduction

In [1], The first-order asymptotic form is obtained and proved for the solution of the first boundary value problem for a quasilinear parabolic equation with small parameters in the derivatives which degenerates into a functional equation. During the construction, smoothing is performed for the corner boundary functions which are unsmooth on a characteristic. In [2], Numerical methods of solving quasilinear heat-conduction equations with a small parameter for the highest-order derivatives with respect to the spatial variables are considered. Nonlinear difference schemes are constructed by the exact difference scheme method. The proposed schemes are uniformly convergent in the small parameter on arbitrary nonuniform grids. Some results on the construction of asymptotics for solutions of singularly perturbed problem with fist-order partial derivatives can be found in [3]. Parallel iterative algorithms combining a time discretization and domain decomposition methods are applied to solution of a singularly perturbed parabolic problem. The domain decomposition methods are based on the Schwarz alternating procedure and time extrapolation. Convergence results for the iterative algorithms are established in [4]. In [5], The first boundary-value problem for the parabolic equation in a rectangle, where $F(u, x, t, \varepsilon)$ is a non-linear function of u, is considered. A formal asymptotic expansion of the solution is constructed. The asymptotic form consists of a regular part, and boundary parts on the sides of the rectangle, and corner parts. The methods of the theory of perturbations and theory of parabolic equations are used. In [6], the researcher considered the Dirichlet problem on a rectangle for singularly perturbed parabolic equations of reaction-diﬀ usion type. The reduced (for ε = 0) equation is an ordinary diﬀ erential equation with respect to the time variable; the singular perturbation parameter ε may take arbitrary values from the half-interval (0,1]. In [7], the authors proposed an algorithm of asymptotic integration of semi-linear initial-boundary-value problems whose minor coefficients are functions oscillating in time with high frequency ω. Recent studies of asymptotic analysis of differential equations involving large high- frequency terms have been carried out in [8, 9]. In [10], the methods proposed in [3] and [7] were combined and an algorithm of asymptotic integration of the initial-boundary-value problem for the heat-conduction equation with nonlinear sources of heat terms oscillating in time with frequency ω^{-1} was developed. Recent studies of asymptotic analysis of differential equations involving large high-frequency terms have been carried out in [11, 12]. For a singularly perturbed first-order partial differential equation, a theorem was proved in [13], on the passage to the limit for the case in which the roots of the degenerate equation intersect and the root intersection line meets the initial segment on which the initial condition is posed. A singularly perturbed system of two second-order differential equations (one rapid and one slow), was considered in [14], which proved the existence of a solution and obtained its asymptotics for the case in which the degenerate equation has two intersecting roots. In [15],considered a new algorithm, based on integral equation formulations, for the solution of constant-coefficient elliptic partial differential equations (PDE) in closed two- dimensional domains with non-smooth boundaries; they focus on cases in which the integral-equation solutions as well as physically meaningful quantities (such as, stresses, electric/magnetic fields, etc.) tend to infinity at singular boundary points (corners). In [16], he considered the second order ordinary differential equations whose coefficients of the unknowns contain smooth and rapidly oscillating summands proportional to the positive powers of the oscillation frequency. The complete asymptotic expansions are constructed and justified of solutions to the Cauchy problem. In [17], the authors considered second-order ordinary differential equation whose coefficients contain smooth and rapidly oscillating summands proportional to the positive powers of the oscillation frequency. In [18], the asymptotic behaviors of a solution of the first boundary value problem for a second order elliptic equation is analysed in the case where a small parameter is involved as a factor multiplying only one of the highest order derivatives and the limit equation is an ordinary differential equation. In [19], for a linear normal system of ordinary differential equations with rapidly oscillating coefficients in a critical case, the existence of a unique periodic solution is proved, its complete asymptotic expansion is constructed and justified, and Lyapunov stability and instability conditions are found. The asymptotic series constructed is shown to converge absolutely and uniformly to the solution. In [20], considered a mixed boundary-initial value problems for a partial differential equation in the critical case for the singular perturbed, and he constructed the asymptotic expansion $u(x,y,t,\varepsilon)$ for the modified problem and proved the uniqueness of the solution. In [21], constructed an asymptotic solution of the first boundary-value problem for a linear singularly perturbed system of hyperbolic partial differential equations with degeneration.

2. Main Problem

We consider the following initial-boundary-value problem depending on the small parameter ε in the domain $(\mathrm{x},\mathrm{t})\in\varpi=(0\le \mathrm{x}\le 1)\mathrm{x}(0\le \mathrm{t}\le \mathrm{T})$.

In this paper we study the following problem:

$$\varepsilon^2\frac{\partial^2 h}{\partial x^2}+\varepsilon c_1(x,t)\frac{\partial h}{\partial t}-c_2(x,t)h=f_0(x,t,\varepsilon)+\varepsilon f_1(x,t,\varepsilon) \qquad (2\text{-}1)$$

$$h\big|_{x=0}=\psi_1(t),\quad h\big|_{x=1}=\psi_2(t),$$

$$h\big|_{t=0}=\varphi_1(x),\quad \left.\frac{\partial h}{\partial t}\right|_{t=0}=\varphi_2(x), \qquad (2\text{-}2)$$

The $c_1(x,t),c_2(x,t),f_0(x,t,\varepsilon),f_1(x,t,\varepsilon),\ \psi_1(t),\psi_2(t),\varphi_1(x), and\ \varphi_2(x)$ functions are continuous and infinitely differentiable with respect to each of their arguments. In what follows we construct and justify an asymptotic expansion of initial-boundary-value type for the solution of the problem (2-1) and (2-2), subject to the following requirements:

I. The functions $\mathrm{c}_1(\mathrm{x},\mathrm{t}) > 0$and $\mathrm{c}_2(\mathrm{x},\mathrm{t})>0,\ \forall(x,t)\in\varpi$.

II. The following compatibility conditions are satisfied:

1. $f(0,0,\varepsilon)=-\mathrm{c}_2(0,0)\varphi_1(0),\varphi_2(0)=0,f(1,0,\varepsilon)=-\mathrm{c}_2(1,0)\varphi_1(1),\varphi_2(1)=0$, ensures that at these points the initial and boundary functions satisfy equation (2-1) for arbitrary ε .
2. $\varphi_1(0)=\psi_1(0),\ \varphi_2(0)=\psi_1'(0),\ \varphi_1(0)=\psi_2(0),\ \varphi_2(0)=\psi_1'(0)$, ensures the compatibility of the initial and boundary functions at the corner point (0, 0) and (1, 0).

III. We shall also expand f in series in powers of ε as

$$\mathbf{f}(x,t,\varepsilon) = \mathbf{f}_0(x,t) + \varepsilon \mathbf{f}_1(x,t) + \frac{\varepsilon^2}{2!}\mathbf{f}_2(x,t) + \ldots.$$

When these conditions are satisfied, there exists a unique classical solution of the problem (2-1), (2-2) for arbitrary $\varepsilon \neq 0$.

3. Algorithm for Construction Asymptotic Form

We seek an asymptotic expansion of the solution of problem (2-1), (2-2) in the form

$$h(x,t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k [u_k(x,t) + r_k(x,\tau) + q_k(\xi,t) + Q_k(\eta,t) + p_k(\xi,\tau) + \overline{P}_k(\eta,\tau)], \qquad (3\text{-}1)$$

Where u_k is the regular part of the asymptotic form, r_k, q_k and Q_k are boundary-layer functions describing boundary layers in the vicinity of the sides t = 0, x = 0 and x = l, and $p_k, \overline{p}_k$ are angular boundary functions which play a role, respectively, in neighbourhoods of the corner points (0,0) and (1,0). Also $\tau = \frac{t}{\varepsilon}$, $\xi = \frac{x}{\varepsilon}$, and $\eta = \frac{1-x}{\varepsilon}$ are boundary layer variables. We substitute series (3-1) into equation (2-1) and equate the coefficients of the same powers of ε on the left and right-hand sides to obtain relations for each of the eight classes of functions in (3-1); thus, we obtain a problem for coefficients in (3-1).

4. Regular Coefficient of the Asymptotic of $u(x,t,\varepsilon)$

We have the problem for the regular coefficient of the asymptotic expansion (3-1), namely $u(x,t,\varepsilon)$, is determined with the aid of the equation:

$$\varepsilon^2 \frac{\partial^2 u}{\partial x^2} + \varepsilon c_1(x,t)\frac{\partial u}{\partial t} - c_2(x,t)u = f_0(x,t,\varepsilon) + \varepsilon f_1(x,t,\varepsilon) \qquad (4\text{-}1)$$

and by using the condition III, we can expend the function $f_0(x,t,\varepsilon)$ and $f_1(x,t,\varepsilon)$.

4.1 Regular Coefficient function of $u_0(x,t)$

Now we find the regular coefficient $u_0(x,t)$ from equation (4-1),

$$-c_2(x,t)u_0 = f_{00}(x,t) \qquad (4\text{-}2)$$

the solution (4-2) has the form,

$$u_0(x,t) = -\frac{f_{00}(x,t)}{c_2(x,t)}.$$

4.2 Regular Coefficient function of $u_2(x,t)$

For the regular coefficient of $u_1(x,t)$ we have the following problem,

$$c_1(x,t)\frac{\partial u_0}{\partial t} - c_2(x,t)u_1 = f_{01}(x,t,\varepsilon) + f_{10}(x,t,\varepsilon) \qquad (4\text{-}3)$$

the solution (4-3) has the form,

$$c_2(x,t)u_1 = -\left(f_{01}(x,t) + f_{10}(x,t) - c_1(x,t)\frac{\partial u_0}{\partial t}\right)$$

$$u_1(x,t) = \frac{-\left(f_{01}(x,t) + f_{10}(x,t) - c_1(x,t)\frac{\partial u_0}{\partial t}\right)}{c_2(x,t)}.$$

4.3 Regular Coefficient function of $u_2(x,t)$

For the regular coefficient of $u_2(x,t)$ we have the following problem,

$$\frac{\partial^2 u_0}{\partial x^2}+c_1(x,t)\frac{\partial u_1}{\partial t}-c_2(x,t)u_2=f_{02}(x,t)+f_{11}(x,t) \qquad (4\text{-}4)$$

the solution (4-4) has the form,

$$c_2(x,t)u_2=-\left(f_{02}(x,t)+f_{11}(x,t)-\frac{\partial^2 u_0}{\partial x^2}-c_1(x,t)\frac{\partial u_1}{\partial t}\right)$$

$$u_2(x,t)=\frac{-\left(f_{02}(x,t)+f_{11}(x,t)-\frac{\partial^2 u_0}{\partial x^2}-c_1(x,t)\frac{\partial u_1}{\partial t}\right)}{c_2(x,t)}$$

4.4 Regular Coefficient function of $u_i(x,t)$

For the regular coefficient of $u_i(x,t), i\geq 3,$ we have the following problem,

$$\frac{\partial^2 u_{i-2}}{\partial x^2}+c_1(x,t)\frac{\partial u_{i-1}}{\partial t}-c_2(x,t)u_i=f_{0i}(x,t)+f_{1i-1}(x,t) \qquad (4\text{-}5)$$

the solution (4-5) has the form,

$$c_2(x,t)u_i=-\left(f_{0i}(x,t)+f_{1i-1}(x,t)-\frac{\partial^2 u_{i-2}}{\partial x^2}-c_1(x,t)\frac{\partial u_{i-1}}{\partial t}\right)$$

$$u_i(x,t)=\frac{-\left(f_{0i}(x,t)+f_{1i-1}(x,t)-\frac{\partial^2 u_{i-2}}{\partial x^2}-c_1(x,t)\frac{\partial u_{i-1}}{\partial t}\right)}{c_2(x,t)}.$$

It is obvious that $u(x,t,\varepsilon)$ satisfies none of the conditions (2-2).

5. Boundary-Layer Coefficient in Neighbourhood of the Initial Time Instant of $r(\mathrm{x},\mathrm{t},\varepsilon)$

We consider problem (2-1), (2-2) in a neighbourhood of the upper boundary (t = 0) of the domain ϖ. Substituting the series (3-1) in (2-1) and equating powers of ε and performing the change of variables $t=\varepsilon\tau$, we have the problem for the boundary-layer function $r(\mathrm{x},\tau,\varepsilon)$.

$$\varepsilon^2\frac{\partial^2 r}{\partial x^2}+\varepsilon c_1(x,\varepsilon\tau)\frac{\partial r}{\partial(\varepsilon\tau)}-c_2(x,\varepsilon\tau)r=f_0(x,\varepsilon\tau,\varepsilon)+\varepsilon f_1(x,\varepsilon\tau,\varepsilon)$$

$$\varepsilon^2\frac{\partial^2 r}{\partial x^2}+c_1(x,\varepsilon\tau)\frac{\partial r}{\partial\tau}-c_2(x,\varepsilon\tau)r=f_0(x,\varepsilon\tau,\varepsilon)+\varepsilon f_1(x,\varepsilon\tau,\varepsilon)$$

with condition when $\tau=0$ we have

$$r(\mathrm{x},0,\varepsilon)=\varphi_1(x)-u(x,0,\varepsilon),\ \frac{1}{\varepsilon}\frac{\partial r}{\partial\tau}(x,0,\varepsilon)=\varphi_2(x)-\frac{\partial u}{\partial\tau}(x,0,\varepsilon) \qquad (5\text{-}1)$$

and by using the condition III, we can expend the function $f_0(x,t,\varepsilon)$ and $f_1(x,t,\varepsilon)$.

5.1 Boundary-Layer Coefficient of $r_0(\mathrm{x},\tau,\varepsilon)$

Now we find the $r_0(\mathrm{x},\tau,\varepsilon)$ from the problem (5-1)

$$c_1(x,0)\frac{\partial r_0}{\partial \tau} - c_2(x,0) r_0 = 0$$

$$r(\mathrm{x},0,\varepsilon) = \varphi_1(x) - u_0(x,0,\varepsilon),\ \frac{\partial r}{\partial \tau}(x,0,\varepsilon) = 0 \qquad (5\text{-}2)$$

from problem (5-2), we obtain $r_0(\mathrm{x},\tau) = 0$, in which x appears as a parameter 0< x< l and $\tau \geq 0$. The root of the corresponding characteristic equation $\lambda(x) = \dfrac{c_2(x,0)}{c_1(x,0)}$, satisfy, by virtue of I, the condition $\lambda(\mathrm{x}) > 0$ for $0 \leq \mathrm{x} \leq 1$.

Hence there is a first degree polynomial $\mu(\tau)$ with positive coefficient, such that the following estimate holds

$$|r_0(x,\tau)| \leq \mu(\tau) e^{\lambda(x)\tau}, 0 \leq x \leq 1, \tau \geq 0. \qquad (5\text{-}3)$$

5.2 Boundary-Layer Coefficient of $r_1(\mathrm{x},\tau,\varepsilon)$

For $r_1(x,\tau)$ we obtain the problems,

$$c_1(x,0)\frac{\partial r_1}{\partial \tau} - c_2(x,0) r_1 = f_{000}(x,\tau)$$

with condition

$$r_1(\mathrm{x},0,\varepsilon) = -u_1(x,0,\varepsilon),\ \frac{\partial r_1}{\partial \tau}(x,0) = -\frac{\partial u_1}{\partial t}(x,0) \qquad (5\text{-}4)$$

the solution of problem (5-4) has the form,

$$r_1(x,\tau) = \beta_1(x,\tau) e^{\lambda(x)\tau}$$

Where $\beta_1(x,\tau)$ and its derivatives of arbitrary order increase as $\tau \to \infty$ no faster than some power of τ .and the estimate of $r_1(x,\tau)$ is

$$|r_1(x,\tau)| \leq \mu(\tau) e^{\lambda(x)\tau}, 0 \leq x \leq 1, \tau \geq 0.$$

5.3 Boundary-Layer Coefficient of $r_2(\mathrm{x},\tau,\varepsilon)$

For $r_2(x,\tau)$ we obtain the problems

$$\frac{\partial^2 r_0}{\partial x^2} + c_1(x,0)\frac{\partial r_2}{\partial \tau} - c_2(x,0) r_2 = f_{001}(x,\tau) + f_{100}(x,\tau)$$

$$c_1(x,0)\frac{\partial r_2}{\partial \tau} - c_2(x,0) r_2 = f_{001}(x,\tau) + f_{100}(x,\tau) - \frac{\partial^2 r_0}{\partial x^2}$$

with condition

$$r_2(\mathrm{x},0,\varepsilon) = -u_2(x,0,\varepsilon),\ \frac{\partial r_2}{\partial \tau}(x,0) = -\frac{\partial u_2}{\partial t}(x,0) \qquad (5\text{-}5)$$

the solution of problem (5-5) has the form ,

$$r_2(x,\tau) = \beta_2(x,\tau) e^{\lambda(x)\tau}$$

Where $\beta_2(x,\tau)$ and its derivatives of arbitrary order increase as $\tau \to \infty$ no faster than some power of τ . So the estimate of $r_2(x,\tau)$ is

$$|r_2(x,\tau)| \leq \mu(\tau) e^{\lambda(x)\tau}, 0 \leq x \leq 1, \tau \geq 0.$$

5.4 Boundary-Layer Coefficient of $r_i(\mathrm{x},\tau,\varepsilon)$

In the same way we can find $r_i(x,\tau), i \geq 3$, we obtain the problems

$$\frac{\partial^2 r_{i-2}}{\partial x^2} + c_1(x,0)\frac{\partial r_i}{\partial \tau} - c_2(x,0)r_i = \rho_{0ij}(x,\tau) + \rho_{1ij}(x,\tau)$$

$$c_1(x,0)\frac{\partial r_i}{\partial \tau} - c_2(x,0)r_i = \rho_{kij}(x,\tau) - \frac{\partial^2 r_{i-2}}{\partial x^2}$$

with condition

$$r_i(\mathrm{x},0) = -u_i(x,), \ \frac{\partial r_i}{\partial \tau}(x,0) = -\frac{\partial u_i}{\partial t}(x,0) \qquad (5\text{-}6)$$

Where the functions $\rho_{nij}(x,\tau), \quad n = 0,1$ are expressed by the coefficient $r_s(x,\tau)$ and $\frac{\partial r_s(x,\tau)}{\partial \tau}, s < i, j$ and where $\rho_{nij}(x,\tau) = 0, \quad n = 0,1$ Similarly, solution of problem (5-6) has the form

$r_i(x,\tau) = \beta_i(x,\tau)e^{\lambda(x)\tau}$,

where $\beta_i(x,\tau)$ and its derivatives of arbitrary order increase as $\tau \to \infty$ no faster than some power of τ and the estimate of $r_i(x,\tau)$ is

$\left|r_i(x,\tau)\right| \leq \mu(\tau)e^{\lambda(x)\tau}, 0 \leq x \leq 1, \tau \geq 0$.

6. Boundary Layer Coefficient in Neighbourhood of the Sides x = 0 and x = 1, of $q_k(\xi,t,\varepsilon)$

The coefficients $q_k(\xi,t,\varepsilon)$, compatible with the regular coefficient, must satisfy the boundary condition for x = 0 and be determined from the problem

$$\varepsilon^2 \frac{\partial^2 q}{\partial(\varepsilon\xi)^2} + \varepsilon c_1(\varepsilon\xi,t)\frac{\partial q}{\partial t} - c_2(\varepsilon\xi,t)q = f_0(\varepsilon\xi,t,\varepsilon) + \varepsilon f_1(\varepsilon\xi,t,\varepsilon)$$

$$\frac{\partial^2 q}{\partial \xi^2} + \varepsilon c_1(\varepsilon\xi,t)\frac{\partial q}{\partial t} - c_2(\varepsilon\xi,t)q = f_0(\varepsilon\xi,t,\varepsilon) + \varepsilon f_1(\varepsilon\xi,t,\varepsilon) \qquad (6\text{-}1)$$

with condition when $\xi = 0$ we have

$$q(0,t) = \psi_1(t) - u(0,t), \ q(\xi,t) \to 0 \ as \ \xi \to \infty \qquad (6\text{-}2)$$

and by using the condition III, we can expend the function $f_0(x,t,\varepsilon)$ and $f_1(x,t,\varepsilon)$.

6. 1 Boundary Layer function in Neighbourhood of $q_0(\xi,t,\varepsilon)$

Now we find the $q_0(\xi,t,\varepsilon)$ from the problem (6-1) and (6-2)

$$\frac{\partial^2 q_0}{\partial \xi^2} - c_2(0,t)q_0 = 0$$

$$q_0(0,t) = \psi_1(t) - u_0(0,t), \ q_0(\xi,t) \to 0 \ as \ \xi \to \infty \qquad (6\text{-}3)$$

in which t appears as a parameter $(0 \leq t \leq T)$, and $\xi \geq 0$. The solution (6-3) has the form

$q_0(\xi,t) = c\,e^{\pm\sqrt{c_2(0,t)}\xi}$, where c is an arbitrary function.

Now we find c by using the condition $q_0(0,t) = \psi_1(t) - u_0(0,t), \ q_0(\xi,t) \to 0 \ as \ \xi \to \infty$

$c = \psi_1(t) - u_0(0,t)$, so the $q_0(\xi,t)$ will be as follows

$q_0(\xi,t) = [\psi_1(t) - u_0(0,t)]e^{\pm\sqrt{c_2(0,t)}\xi}$

and the estimate of $q_0(\xi,t,\varepsilon)$ holds:

$|q_0(\xi,t)| \le \delta(\xi)e^{\pm\sqrt{c_2(0,t)}\xi}, \xi \ge 0, 0 \le t \le T$,

where $\delta(\xi)$ is polynomial with positive coefficient.

6. 2 Boundary Layer function in Neighbourhood of $q_1(\xi,t,\varepsilon)$

For $q_1(\xi,t,\varepsilon)$, we obtain the problems

$$\frac{\partial^2 q_1}{\partial \xi^2} + c_1(0,t)\frac{\partial q_0}{\partial t} - c_2(0,t)q_1 = f_{000}(\xi,t)$$

with condition

$$q_1(0,t) = -u_1(0,t),\; q_1(\xi,t) \to 0 \; as \; \xi \to \infty \qquad (6\text{-}4)$$

The roots of the corresponding characteristic equation (6-4) are

$$\lambda_{1,2}(t) = \frac{-c_1(0,t) \pm \sqrt{c^2{}_1(0,t) + 4c_2(0,t)}}{2},$$

and satisfy, by virtue of I, the condition $\operatorname{Re} \lambda_{1,2}(t) < 0$ for $0 \le t \le T$. Depending on whether $H(t) = c_1^2(0,t) - 4c_2(0,t)$ is greater than, equal to, or less than zero, the solution (6.4) is written differently; however, in each of these cases it can, without difficulty, be written out explicitly and has an exponential estimate.

Let

$$\lambda(t) = \begin{cases} \dfrac{c_1(0,t) - \sqrt{H(t)}}{2}, & if \;\; H(0,t) > 0 \\ \dfrac{c_1(0,t)}{2}, & if \;\; H(0,t) \le 0 \end{cases}$$

it is obvious that when $H(t) \ne 0$ we can find a constant δ, or, when H(t) = 0, we can find a polynomial $\delta(\xi)$ with positive coefficients, such that the following estimate holds:

$|q_1(\xi,t)| \le \delta(\xi)e^{\pm\sqrt{c_2(0,t)}\xi},\; \xi \ge 0,\;\; 0 \le t \le T$.

6. 3 Boundary Layer function in Neighbourhood of $q_2(\xi,t,\varepsilon)$

For $q_2(\xi,t,\varepsilon)$, we obtain the problems

$$\frac{\partial^2 q_2}{\partial \xi^2} + c_1(0,t)\frac{\partial q_1}{\partial t} - c_2(0,t)q_2 = f_{001}(\xi,t) + f_{100}(\xi,t)$$

with condition

$$q_2(0,t) = -u_2(0,t),\; q_2(\xi,t) \to 0 \; as \; \xi \to \infty \qquad (6\text{-}5)$$

in the same case we find the roots of problems (6-5) and we find the estimate of $q_2(\xi,t,\varepsilon)$ is

$|q_2(\xi,t)| \le \delta(\xi)e^{\pm\sqrt{c_2(0,t)}\xi}, \xi \ge 0, 0 \le t \le T$.

6. 4 Boundary Layer function in Neighbourhood of $q_i(\xi,t)$

In the same way we can find $q_i(\xi,t), i \ge 3$, we obtain the problems

$$\frac{\partial^2 q_i}{\partial \xi^2} + c_1(0,t)\frac{\partial q_{i-1}}{\partial t} - c_2(0,t)q_i = \Delta_{0ij}(\xi,t) + \Delta_{1ij}(\xi,t)$$

$q_i(0,t) = -u_i(0,t),\ q_i(\xi,t) \to 0 \ as\ \xi \to \infty$ (6-6)

Where $\Delta_{nij}(\xi,t),\ n = 0,1$ are known functions, and have self similarly roots above and the estimate of $q_i(\xi,t)$ has form

$\left|q_i(\xi,t)\right| \le \delta(\xi) e^{\pm\sqrt{c_2(0,t)}\xi}, \xi \ge 0, 0 \le t \le T$, (6-7)

where $\delta(\xi)$ is polynomial with positive coefficient.

7. Boundary Layer Coefficient in Neighbourhood of the Sides x = 0 and x = 1, of $Q_k(\eta,t,\varepsilon)$

The coefficients $Q_k(\eta,t,\varepsilon)$, compatible with the regular coefficient, must satisfy the boundary condition for x = 0, also we performing the change of $x = 1-\varepsilon\eta$, and be determined from the problem,

$$\varepsilon^2 \frac{\partial^2 q}{\partial(1-\varepsilon\eta)^2} + \varepsilon c_1(1-\varepsilon\eta,t)\frac{\partial q}{\partial t} - c_2(1-\varepsilon\eta,t)q = f_0(1-\varepsilon\eta,t,\varepsilon) + \varepsilon f_1(1-\varepsilon\eta,t,\varepsilon)$$

$$\frac{\partial^2 Q}{\partial \eta^2} + \varepsilon c_1(1-\varepsilon\eta,t)\frac{\partial Q}{\partial t} - c_2(1-\varepsilon\eta,t)Q = f_0(1-\varepsilon\eta,t,\varepsilon) + \varepsilon f_1(1-\varepsilon\eta,t,\varepsilon) \quad (7\text{-}1)$$

with condition when $\eta = 0$ we have

$Q(0,t) = \psi_1(t) - u(0,t),\ Q(\eta,t) \to 0 \ as\ \eta \to \infty$ (7-2)

where $\delta(\xi)$ is polynomial with positive coefficient

and by using the condition III, we can expend the function $f_0(1-\varepsilon\eta,t,\varepsilon)$ and $f_1(1-\varepsilon\eta,t,\varepsilon)$.

7. 1 Boundary Layer function in Neighbourhood of $Q_0(\eta,t,\varepsilon)$

Now we find the $Q_0(\eta,t,\varepsilon)$ from the problem (7-1) and (7-2)

$$\frac{\partial^2 Q_0}{\partial \eta^2} - c_2(0,t)Q_0 = 0$$

with condition

$Q_0(0,t) = \psi_1(t) - u_0(0,t),\ Q_0(\xi,t) \to 0 \ as\ \eta \to \infty$ (7-3)

in which t appears as a parameter $(0 \le t \le T)$, and $\eta \ge 0$.. The solution (7.3) has the form

$Q_0(\eta,t) = c\, e^{\pm\sqrt{c_2(0,t)}\eta}$, where c is an arbitrary function.

Now we find c by using the condition $Q_0(0,t) = \psi_1(t) - u_0(0,t),\ Q_0(\xi,t) \to 0 \ as\ \eta \to \infty$

$c = \psi_1(t) - u_0(0,t)$, so the $Q_0(\xi,t)$ has been

$Q_0(\eta,t) = [\psi_1(t) - u_0(0,t)]e^{\pm\sqrt{c_2(0,t)}\eta}$.

and the estimate of $Q_0(\eta,t,\varepsilon)$ holds:

$\left|Q_0(\xi,t)\right| \le \delta(\eta) e^{\pm\sqrt{c_2(0,t)}\eta}, \eta \ge 0, 0 \le t \le T$,

where $\delta(\eta)$ is polynomial with positive coefficient.

7. 2 Boundary Layer function in Neighbourhood of $Q_1(\eta,t,\varepsilon)$

For $Q_1(\eta,t,\varepsilon)$, we obtain the problems

$$\frac{\partial^2 Q_1}{\partial \eta^2} + c_1(0,t)\frac{\partial Q_0}{\partial t} - c_2(0,t)Q_1 = -f_{000}(\eta,t)$$

with condition

$Q_1(0,t) = -u_1(0,t),\ Q_1(\eta,t) \to 0 \ as\ \eta \to \infty$ (7-4) the roots of the corresponding characteristic equation (7-4) are

$$\lambda_{1,2}(t)=\frac{-c_1(0,t)\pm\sqrt{c^2{}_1(0,t)+4c_2(0,t)}}{2},$$

and satisfy, by virtue of I, the condition $\operatorname{Re}\lambda_{1,2}(t)<0$ for $0 \le t \le T$. Depending on whether H(t) = c_1^2 (0,t)-4c_2 (0,t)is greater than, equal to, or less than zero, the solution (7-4) is written differently; however, in each of these cases it can, without difficulty, be written out explicitly and has an exponential estimate.

Let

$$\lambda(t)=\begin{cases}\dfrac{c_1(0,t)-\sqrt{H(t)}}{2}, & if\ H(0,t)>0\\[2mm] \dfrac{c_1(0,t)}{2}, & if\ H(0,t)\le 0\end{cases}$$

It is obvious that when $\mathrm{H}(\mathrm{t})\neq 0$ we can find a constant δ, or, when H(t) = 0, we can find a polynomial $\delta(\eta)$ with positive coefficients, such that the following estimate holds:

$$\left|Q_1(\eta,t)\right|\le\delta(\eta)e^{\pm\sqrt{c_2(0,t)}\eta},\eta\ge 0,0\le t\le T.$$

7. 3 Boundary Layer function in Neighbourhood of $Q_2(\eta,t,\varepsilon)$

For $Q_2(\eta,t,\varepsilon)$, we obtain the problems

$$\frac{\partial^2 Q_2}{\partial\eta^2}+c_1(0,t)\frac{\partial Q_1}{\partial t}-c_2(0,t)Q_2=-f_{001}(\eta,t)-f_{100}(\eta,t)$$

$$Q_2(0,t)=-u_2(0,t),\ Q_2(\eta,t)\to 0\ as\ \eta\to\infty \qquad (7\text{-}5)$$

in the same case we find the roots of problems (6-5) and we find the estimate of $Q_2(\eta,t,\varepsilon)$ has form

$$\left|Q_2(\eta,t)\right|\le\delta(\eta)e^{\pm\sqrt{c_2(0,t)}\eta},\eta\ge 0,0\le t\le T.$$

where $\delta(\eta)$ is polynomial with positive coefficient.

7. 4 Boundary Layer function in Neighbourhood of $Q_i(\eta,t)$

In the same way we can find $Q_i(\eta,t), i\ge 3,$ we obtain the problems

$$\frac{\partial^2 Q_i}{\partial\eta^2}+c_1(0,t)\frac{\partial Q_{i-1}}{\partial t}-c_2(0,t)Q_i=\Delta_{0ij}(\eta,t)+\Delta_{1ij}(\eta,t)$$

with condition

$$Q_i(0,t)=-u_i(0,t),\ Q_i(\eta,t)\to 0\ as\ \eta\to\infty \qquad (7\text{-}6)$$

Where $\Delta_{nij}(\eta,t),\ n=0,1$ are known functions, and have self similarly roots above. So the estimate of $Q_i(\eta,t)$ is

$$\left|Q_i(\eta,t)\right|\le\delta(\eta)e^{\pm\sqrt{c_2(0,t)}\eta},\eta\ge 0,0\le t\le T, \qquad (7\text{-}7)$$

where $\delta(\eta)$ is polynomial with positive coefficient.

8. The Angular Boundary Coefficients $p_k(\xi,\tau,\varepsilon)$

In this section, we study the angular boundary function $p_k(\xi,\tau,\varepsilon)$ serves to eliminate these residuals in a neighbourhood of the point (0,0).

For $p_k(\xi,\tau,\varepsilon)$, we consider problem (2-1) a neighbourhood of the point (0,0) of the domain ϖ and perform the change of variables $x=\varepsilon\xi$ and $t=\varepsilon\tau$; we get the problem

$$\varepsilon^2 \frac{\partial^2 p}{\partial(\varepsilon\xi)^2}+\varepsilon c_1(\varepsilon\xi,\varepsilon\tau)\frac{\partial p}{\partial(\varepsilon\tau)}-c_2(\varepsilon\xi,\varepsilon\tau)p=f_0(\varepsilon\xi,\varepsilon\tau,\varepsilon)+\varepsilon f_1(\varepsilon\xi,\varepsilon\tau,\varepsilon)$$

$$\frac{\partial^2 p}{\partial\xi^2}+c_1(\varepsilon\xi,\varepsilon\tau)\frac{\partial p}{\partial\tau}-c_2(\varepsilon\xi,\varepsilon\tau)p=f_0(\varepsilon\xi,\varepsilon\tau,\varepsilon)+\varepsilon f_1(\varepsilon\xi,\varepsilon\tau,\varepsilon) \qquad (8\text{-}1)$$

with condition when $\tau=0$ we have

$$p(\xi,0,\varepsilon)=-q(\xi,0,\varepsilon), \qquad \frac{\partial p(\xi,0,\varepsilon)}{\partial\tau}=-\frac{\partial q(\xi,0,\varepsilon)}{\partial t}$$

and when $\xi=0$ we have

$$p(0,\tau,\varepsilon)=-r(0,\tau,\varepsilon) \qquad (8\text{-}2)$$

and by using the condition III, we can expend the function $f_0(\varepsilon\xi,\varepsilon\tau,\varepsilon)$ and $f_1(\varepsilon\xi,\varepsilon\tau,\varepsilon)$.

8.1 The Angular Boundary Coefficients of $p_0(\xi,\tau,\varepsilon)$

From equations (8-1) and (8-2), we have the problem of $p_0(\xi,\tau,\varepsilon)$

$$\frac{\partial^2 p_0}{\partial\xi^2}+c_1(0,0)\frac{\partial p_0}{\partial\tau}-c_2(0,0)p_0=0 \qquad \xi>0,\ \tau>0,$$

$$p_0(\xi,0)=-q_0(\xi,0), \qquad \frac{\partial p_0(\xi,0)}{\partial\tau}=-\varepsilon\frac{\partial q_0(\xi,0)}{\partial t}$$

$$p_0(0,\tau)=-r_0(0,\tau)$$

Since $q_0(\xi,)=[\psi_1(0)-u_0(0,0)]e^{\pm\sqrt{c_2(0,t)}\xi}\equiv 0$, we have $p_0(\xi,0)=0$, and, in exactly the same way $r_0(0,\tau)\equiv 0$, we have $p_0(0,\tau)=0$, in this case we conclude the $p_0(\xi,\tau)\equiv 0$.

8.2 The Angular Boundary Coefficients of $p_1(\xi,\tau,\varepsilon)$,

For the function $p_1(\xi,\tau,\varepsilon)$, we have the following problems

$$\frac{\partial^2 p_1}{\partial\xi^2}+c_1(0,0)\frac{\partial p_1}{\partial\tau}-c_2(0,0)p_1=f_{000}(\xi,\tau)$$

$$p_1(\xi,0)=-q_1(\xi,0), \qquad \frac{\partial p_1(\xi,0)}{\partial\tau}=-\frac{\partial q_1(\xi,0)}{\partial t}$$

$$p_1(0,\tau)=-r_1(0,\tau)$$

for the angular boundary functions $p_1(\xi,\tau,\varepsilon)$, we have the following type of exponential estimate holds :

$$|p_1(\xi,\tau)|\le\begin{cases}\delta(\xi)e^{\alpha\xi}, & \xi\ge\tau\\ \mu(\tau)e^{\beta\tau}, & \xi\le\tau\end{cases}$$

$\alpha>0,\beta>0,$ the polynomials $\delta(\xi)$ and $\mu(\tau)$ are positive coefficients.

8.3 The Angular Boundary Coefficients of $p_2(\xi,\tau,\varepsilon)$

For the function $p_2(\xi,\tau,\varepsilon)$, we have the following problems

$$\frac{\partial^2 p_2}{\partial \xi^2}+c_1(0,0)\frac{\partial p_2}{\partial \tau}-c_2(0,0)p_2=f_{001}(\xi,\tau)+f_{100}(\xi,\tau)$$

$$p_2(\xi,0)=-q_2(\xi,0),\qquad \frac{\partial p_2(\xi,0)}{\partial \tau}=-\frac{\partial q_2(\xi,0)}{\partial t}$$

$$p_2(0,\tau)=-r_2(0,\tau)$$

for the angular boundary functions $p_2(\xi,\tau,\varepsilon)$, we have the following type of exponential estimate holds :

$$|p_2(\xi,\tau)|\leq\begin{cases}\delta(\xi)e^{\alpha\xi}, & \xi\geq\tau\\ \mu(\tau)e^{\beta\tau}, & \xi\leq\tau\end{cases}$$

$\alpha>0,\beta>0,$ the polynomials $\delta(\xi)$ and $\mu(\tau)$ are positive coefficients.

8.4 The Angular Boundary Coefficients of $p_i(\xi,\tau,\varepsilon)$

In the same way we can find $p_i(\xi,\tau,\varepsilon), i\geq 3$ we have the following problems

$$\frac{\partial^2 p_i}{\partial \xi^2}+c_1(0,0)\frac{\partial p_i}{\partial \tau}-c_2(0,0)p_i=\sigma_{0ij}(\xi,\tau)+\sigma_{1ij}(\xi,\tau)$$

$$p_i(\xi,0)=-q_i(\xi,0),\qquad \frac{\partial p_i(\xi,0)}{\partial \tau}=-\frac{\partial q_i(\xi,0)}{\partial t}$$

$$p_i(0,\tau)=-r_i(0,\tau)$$

where $\sigma_{nij}(\xi,\tau),\ n=0,1$ are known functions. For the angular boundary functions $p_i(\xi,\tau),$ the following type of exponential estimate holds :

$$|p_i(\xi,\tau)|\leq\begin{cases}\delta(\xi)e^{\alpha\xi}, & \xi\geq\tau\\ \mu(\tau)e^{\beta\tau}, & \xi\leq\tau\end{cases}$$

$\alpha>0,\beta>0,$ the polynomials $\delta(\xi)$ and $\mu(\tau)$ are positive coefficients.

9. The Angular Boundary Coefficients $\overline{p_k}(\eta,\tau,\varepsilon)$

In this section, we study the angular boundary function $\overline{p_k}(\eta,\tau,\varepsilon)$ serves to eliminate these residuals in a neighbourhood of the point (1,0).

For $\overline{p_k}(\eta,\tau,\varepsilon)$, we consider problem (2-1) a neighbourhood of the point (1,0) of the domain ϖ and perform the change of variables $x=1-\varepsilon\eta$ and $t=\varepsilon\tau$; we get the problem

$$\frac{\partial^2 \overline{p}}{\partial \eta^2}+c_1(1-\varepsilon\eta,\varepsilon\tau)\frac{\partial \overline{p}}{\partial \tau}-c_2(1-\varepsilon\eta,\varepsilon\tau)\overline{p}=f_0(1-\varepsilon\eta,\varepsilon\tau,\varepsilon)+\varepsilon f_1(1-\varepsilon\eta,\varepsilon\tau,\varepsilon) \qquad (9\text{-}1)$$

with condition when $\tau=0$ we have

$$\overline{p}_k(\eta,0,\varepsilon)=-Q_k(\xi,0,\varepsilon),\qquad \frac{\partial \overline{p}_k(\eta,0,\varepsilon)}{\partial \tau}=-\frac{\partial Q_k(\xi,0,\varepsilon)}{\partial t}$$

and when $\eta=0$ we have

$$\overline{p}_k(0,\tau,\varepsilon)=-r_k(0,\tau,\varepsilon) \qquad (9\text{-}2)$$

and by using the condition III, we can expend the function $f_0(1-\varepsilon\eta,\varepsilon\tau,\varepsilon)$ and $f_1(1-\varepsilon\eta,\varepsilon\tau,\varepsilon)$.

9.1 The Angular Boundary function $\overline{p}_0(\eta,\tau,\varepsilon)$

From equations (9-1) and (9-2), we have for $\overline{p}_0(\eta,\tau,\varepsilon)$ the problem

$$\frac{\partial^2 \overline{p}_0}{\partial \eta^2} + c_1(0,0)\frac{\partial \overline{p}_0}{\partial \tau} - c_2(0,0)\overline{p}_0 = 0 \qquad \eta > 0,\ \tau > 0,$$

$$\overline{p}_0(\eta,0) = -Q_0(\eta,0), \qquad \frac{\partial \overline{p}_0(\eta,0)}{\partial \tau} = -\varepsilon\frac{\partial Q_0(\eta,0)}{\partial t}$$

$$\overline{p}_0(0,\tau) = -r_0(0,\tau)$$

Since $Q_0(\eta) = [\psi_1(0) - u_0(0,0)]e^{\pm\sqrt{c_2(0,t)}\eta} \equiv 0$, we have $\overline{p}_0(\eta,0) = 0$, and, in exactly the same way $r_0(0,\tau) \equiv 0$, we have $\overline{p}_0(0,\tau) = 0$, in this case we conclude the $\overline{p}_0(\eta,\tau) \equiv 0$.

9.2 The Angular Boundary function $\overline{p}_1(\eta,\tau,\varepsilon)$

For the function $\overline{p}_1(\eta,\tau,\varepsilon)$, we have the following problems

$$\frac{\partial^2 \overline{p}_1}{\partial \xi^2} + c_1(0,0)\frac{\partial \overline{p}_1}{\partial \tau} - c_2(0,0)\overline{p}_1 = -f_{000}(\eta,\tau)$$

$$\overline{p}_1(\eta,0) = -Q_1(\eta,0), \qquad \frac{\partial \overline{p}_1(\eta,0)}{\partial \tau} = -\frac{\partial Q_1(\eta,0)}{\partial t}$$

$$p_1(0,\tau) = -r_1(0,\tau)$$

For the angular boundary functions $\overline{p}_1(\eta,\tau,\varepsilon)$, we have the following type of exponential estimate holds :

$$|p_1(\eta,\tau)| \le \begin{cases} \delta(\eta)e^{\alpha\eta}, & \eta \ge \tau \\ \mu(\tau)e^{\beta\tau}, & \eta \le \tau \end{cases}$$

$\alpha > 0, \beta > 0,$ the polynomials $\delta(\eta)$ and $\mu(\tau)$ are positive coefficients.

9.3 The Angular Boundary function $\overline{p}_2(\eta,\tau,\varepsilon)$

For the function $\overline{p}_2(\eta,\tau,\varepsilon)$, we have the following problems

$$\frac{\partial^2 \overline{p}_2}{\partial \xi^2} + c_1(0,0)\frac{\partial \overline{p}_2}{\partial \tau} - c_2(0,0)\overline{p}_2 = f_{001}(\eta,\tau) + f_{100}(\eta,\tau)$$

$$\overline{p}_2(\eta,0) = -Q_2(\eta,0), \qquad \frac{\partial \overline{p}_2(\eta,0)}{\partial \tau} = -\frac{\partial Q_2(\eta,0)}{\partial t}$$

$$\overline{p}_2(0,\tau) = -r_2(0,\tau)$$

for the angular boundary functions $\overline{p}_2(\eta,\tau,\varepsilon)$, we have the following type of exponential estimate holds :

$$|\overline{p}_2(\eta,\tau)| \le \begin{cases} \delta(\eta)e^{\alpha\eta}, & \eta \ge \tau \\ \mu(\tau)e^{\beta\tau}, & \xi \le \tau \end{cases}$$

$\alpha > 0, \beta > 0,$ the polynomials $\delta(\eta)$ and $\mu(\tau)$ are positive coefficients.

9.4 The Angular Boundary function $\overline{p}_i(\eta,\tau,\varepsilon)$

In the same way we can find $\overline{p}_i(\eta,\tau,\varepsilon), i \ge 3$ we have the following problems

$$\frac{\partial^2 \overline{p}_i}{\partial \xi^2} + c_1(0,0)\frac{\partial \overline{p}_i}{\partial \tau} - c_2(0,0)\overline{p}_i = \sigma_{0ij}(\eta,\tau) + \sigma_{1ij}(\eta,\tau)$$

$$\overline{p}_i(\eta,0) = -Q_i(\eta,0), \qquad \frac{\partial p_i(\eta,0)}{\partial \tau} = -\frac{\partial Q_i(\eta,0)}{\partial t}$$

$$\overline{p}_i(0,\tau) = -r_i(0,\tau)$$

where $\sigma_{nij}(\eta,\tau)$, $n = 0,1$ are known functions. For the angular boundary functions $p_i(\eta,\tau)$, the following type of exponential estimate holds :

$$\left|\overline{p}_i(\eta,\tau)\right| \le \begin{cases} \delta(\eta)e^{\alpha\eta}, & \eta \ge \tau \\ \mu(\tau)e^{\beta\tau}, & \eta \le \tau \end{cases}$$

$\alpha > 0, \beta > 0,$ the polynomials $\delta(\eta)$ and $\mu(\tau)$ are positive coefficients.

10. Estimate of the Remainder Terms

We denote by U_n the n[th] partial sums of the series (3-1).

Theorem

The solution $h(x,t,\varepsilon)$ of the problem (2-1) and (2-2) are valid for small ε uniformly in the domain $\varpi = (0 \le x \le 1)\times(0 \le t \le T)$:

$$h(x,t,\varepsilon) - H_n(x,t,\varepsilon) = O(\varepsilon^{n+1}).$$

Proof

Let $w = h - H_{n+2}$. If we substitute $h = H_{n+2} + w$. in problem (2-1) and (2-2), we have, for all remainder terms w, the following problem

$$\varepsilon^2 \frac{\partial^2 w}{\partial x^2} + \varepsilon c_1(x,t)\frac{\partial w}{\partial t} - c_2(x,t)w = W_0(x,t,\varepsilon) + \varepsilon W_1(x,t,\varepsilon)$$

$$w\big|_{x=0} = -\sum_{i=0}^{n+2}\varepsilon^i\left[Q_I(\frac{1}{\varepsilon},t) + \overline{p}_i(\frac{1}{\varepsilon},\frac{t}{\varepsilon})\right] = O\left(e^{\frac{\alpha}{\varepsilon}}\right),\ w\big|_{x=1} = O\left(e^{\frac{\alpha}{\varepsilon}}\right),$$

$$w\big|_{t=0} = 0, \quad \frac{\partial w}{\partial t}\bigg|_{t=0} = -\varepsilon^{n+2}\left[\frac{\partial u_{n+2}}{\partial t}(x.0) + \frac{\partial q_{n+2}}{\partial t}(\xi.0) + \frac{\partial Q_{n+2}}{\partial t}(\eta.0)\right] = O\left(\varepsilon^{n+2}\right),$$

where $\alpha > 0$, $W_0(x,t,\varepsilon) = f_0(x,t,\varepsilon) - \varepsilon^2\frac{\partial^2 H_{n+2}}{\partial x^2} - \varepsilon c_1(x,t)\frac{\partial H_{n+2}}{\partial t} + c_2(x,t)H_{n+2}$

and $W_1(x,t,\varepsilon) = f_1(x,t,\varepsilon)$. we can assume that

$$\frac{\partial w}{\partial t}\bigg|_{x=0} = \frac{\partial w}{\partial t}\bigg|_{x=1} = 0. \qquad (10\text{-}1)$$

We introduce the function

$$A(t,\varepsilon) = \int_0^1\left(\varepsilon^2\frac{\partial w}{\partial x} - 2c_2(x,t)w^2\right)dx.$$

Differentiating $A(t,\varepsilon)$ with respect to t and using the equality

$$\int_0^1 \frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial x\,\partial t} = -\int_0^1\frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial x^2},$$

obtained by integrating by parts, and making use of (8-1), we find

$$\frac{d}{dt}A(t,\varepsilon)=\int_0^1\left[-\varepsilon c_1(x,t)\left(\frac{\partial w}{\partial t}\right)^2+W_0(x,t,\varepsilon)\frac{\partial w}{\partial t}+\varepsilon W_1(x,t,\varepsilon)\frac{\partial w}{\partial t}-\frac{\partial c_2(x,t)}{\partial t}w^2\right]dx$$

$$+\varepsilon c_2(x,t)\left(\frac{\partial w}{\partial t}\right)^2\leq 0,$$

And $\left(W_0(x,t,\varepsilon)+\varepsilon W_1(x,t,\varepsilon)\right)\frac{\partial w}{\partial t}\leq\left(\frac{W_0+\varepsilon W_1}{\varepsilon}\right)^2+\varepsilon\left(\frac{\partial w}{\partial t}\right)^2,$ (10-2)

since, it follows that

$$\frac{d}{dt}A(t,\varepsilon)\leq\int_0^1-\varepsilon\left(\frac{\partial w}{\partial t}\right)^2dx-\max_{\varpi}\left|\frac{1}{c_2(x,t)}\frac{\partial c_2(x,t)}{\partial t}\right|\int_0^1 c_2(x,t)w\,dx+\int_0^1\left(\frac{W_0+\varepsilon W_1}{\varepsilon}\right)^2dx.$$

From above we have the inequality

$$\frac{d}{dt}A(t,\varepsilon)\leq\gamma A(t,\varepsilon)+\Phi(t,\varepsilon),$$

where

$$\gamma=\max\left\{1,-\max_{\varpi}\left|\frac{1}{c_2(x,t)}\frac{\partial c_2(x,t)}{\partial t}\right|\right\}\text{ and }\Phi(t,\varepsilon)=\int_0^1\left(\frac{W_0+\varepsilon W_1}{\varepsilon}\right)^2dx.$$

Integrating this differential inequality

$$A(0,\varepsilon)\leq\int_0^1-\varepsilon\left(\frac{\partial w}{\partial t}(x,0,\varepsilon)\right)^2dx=O(\varepsilon^{n+1}),$$

that means $|W(x,t,\varepsilon)|=O\left(\varepsilon^{n+1}\right)$,

where $W(x,t,\varepsilon)=W_0(x,t,\varepsilon)+\varepsilon W_1(x,t,\varepsilon)$,

since $w=h-H_{n+2}$. it follows that $h-H_{n+2}=O\left(\varepsilon^{n+1}\right)$.

That proof is therefore completed.

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