The Periodic Solution Ordinary Differential Equations $4-^{th}$ order Containing Frequency Terms

Hayder Jabbar Abood
Email:drhayder_jabbar@yahoo.com

Rusul Mhammed Hussien
Email:russll.alshmary@yahoo.com

We consider the problem of periodic solution of ordinary differential equations of arbitrary $4-^{th}$ order with a rapidly oscillating coefficient proportional to the frequency of oscillations. We prove the existence and local uniqueness of solutions close to the corresponding asymptotic solutions of the original and averaged problems with natural additional conditions of smoothness.

1. Introduction

The present paper deals with asymptotic expansions of periodic solutions of $4-^{th}$ order equations and formal asymptotics of such solutions in the case of equations containing terms oscillating in time with frequency $\omega >> 1$ and proportional to the powers ω^2. Solutions of differential equations usually cannot be represented in quadrature rules using elementary or special functions. In connection with this issue for approximate solutions of differential equations, is an attempt to apply approximate methods that include numerical and asymptotic methods. The thesis of the paper deals with the equation containing large high-frequency terms. In [1], studied the differential equation of the forced oscillations of a mechanical system with one degree of freedom is examined in the case when the system's natural frequency is much greater than the external one, it is shown that periodic solutions of such an equation exist, close to the periodic solutions of the corresponding degenerate equation. In [2], they predict the accurate bifurcating periodic solution for a general class of first-order nonlinear delay differential equation with reflectional symmetry by constructing an approximate technique, named residue harmonic balance, this technique combines the features of the homotopy concept with harmonic balance which leads to easy computation and gives accurate prediction on the periodic solution to the desired accuracy. In [3], considered the problem on the periodic solutions of a system of ordinary differential equations of arbitrary order 4 containing terms oscillating at a frequency $\omega >> 1$ with coefficients of the order of ω^2. In [4], an ordinary differential equation which models the forsional motion of a horizontal cross section of a suspension bridge. We use Leray-Schauder degree theory to prove that the undamped equation has multiple periodic weak solutions. In [5], motivated by some relevant physical applications, studied the existence and uniqueness of T-periodic solutions for a second order differential equation with a piecewise constant singularity which changes sign. In [6], studied the existence and asymptotic stability of periodic solutions of the differential equation

x+f(x)x+g(x)=h(x)

where f(x) is positive and g(x) is strictly monotonically increasing and has one or two weak singularities, and the method of proof relies on the construction of positively invariant region of the flux. In [7], the existence and multiplicity of positive periodic solutions for first non-autonomous singular systems are established with superlinearity or sublinearity assumptions at infinity for an appropriately chosen parameter , and the proof of results is based on the Krasnoselskii fixed point theorem in a cone. In [8], studied multiple periodic solutions of the following non-autonomous delay differential equations

$$x'(t) = -f\,(t, x(t-\frac{\pi}{2})) \tag{1.1}$$

The where assume that $x(t)\in R^n, f \in C\left(R\times R^n, R^n\right)$. *where*

$\frac{\pi}{2}$ – periodic with respect to t i.e (f1) f(t,x) is odd with respect to x and

$$f(t,-x)=-f(t,x), \qquad f(t+\frac{\pi}{2},x)=f(t,x), \qquad \forall (t,x)\in R\times R^n;$$

(f2) there exists a continuous differential function F(t,x), which is strictly convex with respect to x uniformly in t, such that f(t,x) is the gradient of F(t,x) with respect to x;

(f3)

$$f(t,x)=A_o x+g(t,x), as\ g(t,x)=0(|x|) \quad \text{uniformly} |x\to 0| \qquad in\ t;$$

$$f(t,x)=A_\infty x+h(t,x),\ as\ h(t,x)=0(|x|) \quad \text{uniformly} |x\to \infty| \qquad \text{in}\ t;$$

where A_O, A_∞ are positive definite constant matrices, by making use of the Clarke dual, they were studied the dual variational function associated with (1.1), which is an indefinite functional. The presence of high-frequency terms, that are big, creates certain challenges. They are usually at the initial stages of application of asymptotic methods. An important asymptotic technique in the study of differential equations with high-frequency terms is the method of averaging in ([9,10]). The paper is devoted to further development of the classical theory of averaging. Note that equations with high-frequency terms frequent various sections of science.

From basic results of classical theory of the averaging method being investigated, the system of ordinary differential equations in the standard from can be represented as follows:

$$\frac{dx}{dt}=f(x,\omega t), \qquad \omega>>1, \qquad (1.2)$$

Where $f(x,\tau)$ has an average over τ

$$\langle f\rangle(x)=\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x,\tau)d\tau.$$

It should be recalled that for ℓ - periodic and τ vector functions $f(x,\tau)$ the average is determined by the formula:

$$\langle f\rangle(x)=\frac{1}{\ell}\int_0^\ell f(x,s)ds.$$

In recent results of numerous studied of the theory of the averaging methods, there has been a movement to new and different classes of equations. Here we mention only a few studies in which the differential equations, rapidly oscillating in time, are proportional to positive powers of high-frequency oscillations. Among these works are studies of (see[11,12]) on high-frequency vibrations of the suspension point of a simple pendulum to its upper position that can become stable. In [13], the authors show that high-compression-tension beams can enhance its stability. In [14], systems of equations with fast and slow variables were studied. In [15], an efficient algorithm of asymptotic integration of some classes of equations with high-frequency members was proposed. Yudovich V.E. (1991) in his lectures on the method of averaging to the Faculty of Mechanics, Rostov State University, noted the relevance of the theory of averaging for differential equations, differential and partial, containing rapidly oscillating terms proportional to positive powers of the frequency of oscillations ω. In his lectures he considers some of these problems. In particular (see [16]), he studied the motion of mechanical systems with constraints and the motion of ideal fluids in high-power fields. They explored the system of equations of type:

$$\frac{dx}{dt} = f\ (x,t,\omega t) + \omega^{\alpha}\varphi(x,t,\omega t), \qquad \alpha > 0, \qquad (1.3)$$

Where the vector function $f\ (x,t,\omega t)$ and $\varphi(x,t,\omega t)$ ℓ-periodic in τ, and φ-has zero mean along with other equations with the specified characteristics. It was noted that $\alpha = \frac{1}{2}$- is the minimum value of the exponent he named in the lectures (first indicator), where the averaged problem for (1.3) may be different from the homogenized problem for (1.2). This is problem (1.3) with crossed a large term, for equations of the second-order

$$x = \Psi(x,x,t,\omega t) + \omega^{\beta}\chi(x,t,\omega\, t), \qquad (1.4)$$

Where $\Psi(x,x,t,\omega t)$ and $\chi(x,t,\omega t)$ are ℓ- periodic in τ, $\langle\chi\rangle = 0,$ with the adjustment rate $\beta = 1.$ Equations (1.3) and (1.4) can be set as a Cauchy problem that is $\ell\omega^{-1}$- periodic on the whole time axis solution. Asymptotic expansions of periodic solutions of second- and third- order equations and formal asymptotic of such solutions in the case of equations of arbitrary order were constructed in [17,18]. Recent studied of asymptotic analysis of differential equations involving large high-frequency terms have been carried out in [19,20]. In the present paper, we continue the line of research initiated in [18], for 4-th order ordinary differential equations and justify the averaging method that has proved the existence and local uniqueness of solutions close to corresponding asymptotic solutions of the original and averaged problems with natural additional conditions of smoothness.

2.problem Formulation

Regard the problem of $2\pi\omega^{-1}$- periodic solution to differential equation 4-th order by the following problem

$$\frac{d^4u}{dt^4} = f_o(u,\omega t) + \omega^2 f_1(u,\omega t). \qquad (2.1)$$

Where ω - big parameter, the functions $f_o(u,\tau)$ and $f_1(u,\tau)$ are define to the set $(u,\tau) \in R \times R$, continuous and infinitely differential with respect to u, and also is 2π-periodic with respect to τ moreover, the mean value of the function $f_1(u,\tau)$ with respect to τ is zero,

$$\langle f_1(u,\tau)\rangle = \frac{1}{2\pi}\int_0^{2\pi} f_1(u,s)\,ds = 0$$

Let $\omega(\tau) = \varphi(u,\tau) - 2\pi$ periodic with respect to τ be a solution with zero mean value of the equation:

$$\frac{d^4w}{d\tau^4} = f_1(u,\tau),$$

With $u \in R.$ such a solution is known to be unique. the equation

$$\frac{d^4v}{dt^4} = \langle f_o(v,\tau) + \frac{\partial f_1(v,\tau)}{\partial v}\phi(v,\tau)\rangle,$$

Is called the averaged equation for (2.1), assuming that it has a stationary solution $v = u_o$ such as for all function $\Phi(u) = \langle f_o(u,\tau) + \frac{\partial f_1(u,\tau)}{\partial u}\phi(u,\tau)\rangle$ is correct equation with $\Phi(u_o) = 0.$ also we will presuppose that the solution u_o -is non singular, that is,

$$\Phi'(u) = \langle\frac{\partial f_o(u,\tau)}{\partial u} + \frac{\partial^2 f_1(u,\tau)}{\partial u^2}\varphi(u,\tau)\rangle + \left\langle \frac{\partial f_1(u_o,\tau)}{\partial u}\frac{\varphi(u_o,\tau)}{\partial u}\right\rangle \neq 0. \qquad (2.2)$$

We consider matrix:

$$p=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Phi'(u_o) & 0 & 0 & 0 \end{pmatrix}_{4\times 4}$$

The equation $|P-\lambda E|=0,$ has 4- difference roots $(-1)^4\lambda^4-\Phi'(u_o)=0, \lambda_i\neq 0, i=1,2,3,4.$ therefore, spectrum matrix e^{tp} is consisted from characteristic value $e^{t\lambda_1}, e^{t\lambda_2}, e^{t\lambda_3}, e^{t\lambda_4}.$ we denoted by t_1 such positive number that is

$\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, \lambda_4 t_4 \neq 2\pi ki, k\in z.$

Recall a definition of "Holder space"

$C_\mu(\lfloor 0,t_1\rfloor), \mu\in(0,1).$

It is constituted from continuous function $u:\lfloor 0,t_1\rfloor\to R$ satisfying condition:

$$\|u\|_{C\mu([o,t_1])}=\sup|u(t)|+\sup\frac{|u(t'')-u(t')|}{|t''-t'|^\mu}<\infty.$$

the following theorem is holds.

Theorem 2.1. there exist positive number w_o and $\delta,$ such that when $w>w_o$ the equation (2.1) has uniquely in ball $\|u(t)-u_o\|_{C_{\mu(R)}}\leq\delta, 2\pi\omega^{-1}$- periodic solution u_ω. In this connection, the equality is valid: $\lim\limits_{\omega\to\infty}\|u_\omega-u_o\|_{C_{\mu(R)}}=0.$

Proof. the proof of the theorem can be divided in to secs.3 and 4.

3.procedure of solving the problem

Lemma 3.1. let $\phi(t), T$ -periodic function. We consider equation

$$\frac{d^4u}{dt^4}-au=\phi(t), \qquad (3.1)$$

Denoted $\lambda^4=a$ where $a=\rho(\cos\theta+i\sin\theta), \quad 0\leq\theta\leq 1$

$$\lambda=\sqrt[4]{\rho}\left(\cos\frac{\theta+2\pi k}{4}+i\sin\frac{\theta+2\pi k}{4}\right), k\in Z.$$

If the solution u(t) of equation (3.1)satisfy condition u(0)=u(T), then it is T-periodic and valid as here represented:

$$\begin{pmatrix} u(t) \\ \frac{du(t)}{dt} \\ \frac{d^2u(t)}{dt^2} \\ \frac{d^3u(t)}{dt^3} \end{pmatrix}=\left[E-e^{TD}\right]^{-1}\int_0^T e^{(T+t-\tau)D}\Psi(\tau)d\tau+\int_0^t e^{(t-\tau)D}\Psi(\tau)d\tau,$$

Where the matrix is

$$D=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\a&0&0&0\end{pmatrix}_{4\times4}\quad \text{and}\quad \Psi(t)=\begin{pmatrix}0\\0\\0\\\phi(t)\end{pmatrix}_{4\times1}$$

proof. lemma (3.1) is presented in section 4.
We handle the proof of (2.1) with the change of variables ,

$$u(t)=v(t)+\omega^{-2}\phi(v,\omega t) \qquad (3.2)$$

Because the equation does not contain big terms. we make use the following construct relation derived from principles of differential equation and a vergent theorem:

$$\frac{du}{dt}=\left(1+\omega^{-2}\frac{\partial\phi(v,\tau)}{\partial v}\right)\frac{dv}{dt}+\omega^{-1}\frac{\partial\phi(v,\tau)}{\partial\tau} \qquad (3.3)$$

$$\frac{d^2u}{dt^2}=\left(1+\omega^{-2}\frac{\partial\phi(v,\tau)}{\partial v}\right)\frac{d^2v}{dt^2}+\omega^{-2}\frac{\partial^2\phi(v,\tau)}{\partial v^2}\left(\frac{dv}{dt}\right)^2+2\omega^{-1}\frac{\partial^2\phi(v,\tau)}{\partial v\,\partial\tau}\frac{dv}{dt}+\frac{\partial^2\phi(v,\tau)}{\partial\tau^2}$$

$\vdots$

$$\frac{d^4u}{dt^4}=\left(1+\omega^{-2}\frac{\partial\phi(v,\tau)}{\partial v}\right)\frac{d^4v}{dt^4}+4\left[\omega^{-2}\frac{\partial^2\phi(v,\tau)}{\partial v^2}\frac{dv}{dt}+\omega^{-1}\frac{\partial^2\phi(v,\tau)}{\partial v\,\partial\tau}\right]\frac{d^3v}{dt^3}+\sum_{m=2}^{2}\Psi_2(v,\tau)\ \text{where:}$$

$$+4\omega\frac{\partial^4\phi}{\partial v\,\partial\tau^3}\frac{dv}{dt}+\omega^2\frac{\partial^4\phi}{\partial\tau^4}$$

$$(3.4)$$

$$\Psi_2(v,\tau)=\left(\begin{matrix}2\\4\end{matrix}\left[\frac{\partial^4\phi(v,\tau)}{\partial v^2\partial\tau^2}\left(\frac{dv}{dt}\right)^2+\sum_{s=2}^{2}\frac{\partial^{5-s}\phi(v,\tau)}{\partial v^{3-s}\partial\tau^2}\times\left(\frac{dv}{dt}\right)^{2-s}\frac{d^sv}{dt^s}\right]\right),$$

Suppose

$$f_o(v+\Delta,\tau)=f_o(v,\tau)+\frac{\partial f_o(v+\theta\Delta,\tau)}{\partial v}\Delta, and$$

$$\omega^2 f_1(v+\Delta,\tau)=\omega^2\left[f_1(v,\tau)+\frac{\partial f_1(v+\theta\Delta,\tau)}{\partial v}\Delta+\frac{1}{2}\frac{\partial^2 f_1(v+\theta\Delta,\tau)}{\partial v^2}\Delta^2\right]$$

where :

$$\Delta=\omega^{-2}\phi(v(t),\tau),\qquad 0\le\theta\le1$$

substituting equation (3.2) in to equation (2.1) and applying (3.2) and (3.3) we have :

$$\left(1+\omega^{-2}\frac{\partial\phi}{\partial v}\right)\frac{d^4v}{dt^4}+\omega^2\frac{\partial^4\phi}{\partial\tau^4}=f_o(v,\tau)+\frac{\partial f_1(v,\tau)}{\partial v}\phi(v,\tau)+\omega^2 f_1(v,\tau)+\frac{\partial f_o(v+\theta\Delta,\tau)}{\partial v}\Delta$$
$$+\frac{1}{2}\omega^{-2}\frac{\partial^2 f_1(v+\theta\Delta,\tau)}{\partial v^2}-4\left[\omega^{-2}\frac{\partial^2\phi}{\partial v^2}\frac{dv}{dt}+\omega^{-1}\frac{\partial^2\phi}{\partial v\,\partial\tau}\right]\frac{d^3v}{dt^3}-\Psi_2(v,\tau)+4\omega\frac{\partial^4\phi}{\partial v\,\partial\tau^3}\frac{dv}{dt}.$$

We denoted $J=1+\gamma$,where $\gamma=\omega^{-2}\frac{\partial\phi(v,\tau)}{\partial v}$,from above ,we have

$$\frac{d^4v}{dt^4}=f_o(v,\tau)+\frac{\partial f_1(v,\tau)}{\partial v}\phi(v,\tau)+M_o(v,\tau)-\Psi_2(v,\tau)-4\omega\frac{\partial^4\phi}{\partial v\,\partial\tau^3}\frac{dv}{dt}. \tag{3.5}$$

Where:

$$M_o(v,\tau)=J^{-1}\left\{\begin{array}{l}\frac{\partial f_o(v+\theta\Delta,\tau)}{\partial v}\Delta+2\omega^{-2}\frac{\partial^2 f_1(v+\theta\Delta,\tau)}{\partial v^2}\phi^2(v,\tau)\\ -4\left[\omega^{-2}\frac{\partial^2\phi}{\partial v^2}\frac{dv}{dt}+\omega^{-1}\frac{\partial^2\phi}{\partial v\,\partial\tau}\right]\frac{d^3v}{dt^3}-\ldots\end{array}\right\}+\left(-\gamma+\gamma^2-\gamma^3+\ldots\right)$$
$$\left\{f_o(v,\tau)+\frac{\partial f_1(v,\tau)}{\partial v}\phi(v,\tau)-\Psi_2(v,\tau)-4\omega\frac{\partial^4\phi}{\partial v\,\partial\tau^3}\frac{dv}{dt}\right\}.$$

In above equation degree ,we now have the terms ω .In the following step , we convert to an equation with degree, ω^0 and coefficient

$$\left[\frac{\partial^4\varphi}{\partial v^2\partial\tau^2}\left(\frac{dv}{dt}\right)^2+\frac{\partial^3\varphi}{\partial v\,\partial\tau^2}\frac{d^2v}{dt^2}\right]$$

We prove that 2-second step , gives the equation in second with respect ω .the coefficient of $\Psi_2(v,\tau)$ is :

$$\Psi_2(v,\tau)=\binom{2}{4}\left[\frac{\partial^4\varphi(v,\tau)}{\partial v^2\partial\tau^2}\left(\frac{dv}{dt}\right)^2+\sum_{s=2}^{2}\binom{2}{4}\frac{\partial^{5-s}\varphi(v,\tau)}{\partial v^{3-s}\partial\tau^2}\times\left(\frac{dv}{dt}\right)^{2-s}\frac{d^sv}{dt^s}\right]$$

When m=2 our statement is therefore correct . we have that it is correct with m=k, meaning

$$\frac{d^2v}{dt^2}=f_o(v,\tau)+\frac{\partial f_1(v,\tau)}{\partial v}\phi(v,\tau)+M_2(v,\tau)+\cdots-\Psi_2(v,\tau).$$

where :

$$\Psi_2(v,\tau)=\binom{2}{4}\left[\frac{\partial^4\varphi(v,\tau)}{\partial v^2\partial\tau^2}\left(\frac{dv}{dt}\right)^2+\sum_{s=2}^{2}\binom{2}{4}\frac{\partial^{5-s}\varphi(v,\tau)}{\partial v^{3-s}\partial\tau^2}\times\left(\frac{dv}{dt}\right)^{2-s}\frac{d^sv}{dt^s}\right]$$

from the equation (3.5) we get the system :

$$\left\{\begin{array}{l}\frac{dv}{dt}=x_1\\ \frac{d^3x_1}{dt^3}=f_o(v,\tau)+\frac{\partial f_1(v,\tau)}{\partial v}\phi(v,\tau)+M_o(v,\tau)-\cdots-4\omega\frac{\partial^4\phi}{\partial v\,\partial\tau^3}x_1 \qquad (3.6)\end{array}\right\}$$

We make a change of variables :

$$x_1 = w_1 - 4\omega^{-2}\frac{\partial\phi(v,\tau)}{\partial v}w_1$$

$$\frac{dx_1}{dt} = \left(1-\omega^{-2}\frac{\partial\phi(v,\tau)}{\partial v}\right)\frac{dw_1}{dt} - 4\omega^{-2}\frac{\partial^2\phi(v,\tau)}{\partial v^2}\frac{dv}{dt}w_1 + 4\omega^{-1}\frac{\partial^2\phi(v,\tau)}{\partial v\,\partial\tau}w_1$$

$$\vdots$$

$$\frac{d^3x_1}{dt^3} = \left(1-4\omega^{-2}\frac{\partial\phi(v,\tau)}{\partial v}\right)\frac{d^3w_1}{dt^3} - 4\left[\omega^{-2}\frac{\partial^2\phi(v,\tau)}{\partial v^2}\frac{dv}{dt} + \omega^{-1}\frac{\partial^2\phi(v,\tau)}{\partial v\,\partial\tau}\right]\times\frac{d^2w_1}{dt^2} - \cdots$$

$$-4\omega\frac{\partial^4\phi(v,\tau)}{\partial v\,\partial\tau^3}w_1. \qquad (3.7)$$

If we substitute (3.6) in the equation (3.7) we ge we denote $J_1 = 1+\gamma_1, where \;\; \gamma_1 = -4\omega^2\frac{\partial\phi(v,\tau)}{\partial v}$ from here we have :

$$\frac{d^3w_1}{dt^3} = f_o(v,\tau) + \frac{\partial f_1(v,\tau)}{\partial v}\phi(v,\tau) + M_1(v,\tau) - \cdots - 6\left[\frac{\partial^4\phi(v,\tau)}{\partial v^2\partial\tau^2}w_1^{\,2} + \frac{\partial^5\phi(v,\tau)}{\partial v\,\partial\tau^2}\frac{dw_1}{dt}\right] \qquad (3.8)$$

Where:

$$M_1(v,\tau) = J_1^{-1}\left\{M_o(v,\tau) + 4\left[\omega^{-2}\frac{\partial^2\phi}{\partial v^2}\frac{dv}{dt} + \omega^{-1}\frac{\partial^2\phi}{\partial v\,\partial\tau}\right]\frac{d^2w_1}{dt^2} + \cdots\right\} +$$

$$\left(-\gamma_1 + \gamma_1^{\,2} - \gamma_1^{\,3} + \cdots\right)\left(f_o(v,\tau) + \frac{\partial f_1(v,\tau)}{\partial v}\phi(v,\tau)\right.$$

$$\left. - \cdots - 6\omega\left[\frac{\partial^4\phi}{\partial v^2\partial\tau^2}\left(\frac{dv}{dt}\right)^2 + \frac{\partial^3\phi}{\partial v\,\partial\tau^2}\frac{d^2v}{dt^2}\right]\right).$$

From above equation , can therefore write the system :

$$\left\{\begin{array}{l} \dfrac{dw_1}{dt} = x_2 \\ \dfrac{d^2x_2}{dt^2} = f_o(v,\tau) + \dfrac{\partial f_1(v,\tau)}{\partial v}\phi(v,\tau) + M_1(v,\tau) - \cdots - 6\left[\dfrac{\partial^4\phi(v,\tau)}{\partial v^2\partial\tau^2}\omega_1^{\,2} + \dfrac{\partial^3\phi(v,\tau)}{\partial v\,\partial\tau^2}x_2\right] \end{array}\right\} \qquad (3.9)$$

Further we change the variable according to above reduced plan . In this case , the change variable for x_2 is :

$$x_2 = w_2 - 6\omega^{-2}\left[\frac{\partial^2\phi}{\partial v^2}w_1^{\,2} + \frac{\partial\phi}{\partial v}w_2\right]$$

Now , for 2-step, we change the variable as follows :

$$x_2 = w_2 - \binom{4}{2}\omega^{-2}\left[\frac{\partial^2\phi}{\partial v^2}w_1^{\ 2} + \cdots + \frac{\partial\phi}{\partial v}w_2\right]$$

And hence we have the equation :

$$\frac{d^2 w_2}{dt^2} = f_o(v,\tau) + \frac{\partial f_1(v,\tau)}{\partial v}\phi(v,\tau) + M_2(v,\tau).$$

The last equation we can write in the form :

$$\frac{d^2 w_2}{dt^2} = \varphi'(u_o)\,v + f_o(v,\tau) + \frac{\partial f_1(v,\tau)}{\partial v}\phi(v,\tau) + M_2(v,\tau) - \varphi'(u_o)\,v$$

In this step , taking the change variable in to account we have the following system:

$$\left(\begin{aligned} &\frac{dv}{dt} = w_1 - 4\omega^{-2}\frac{\partial\phi(v,\tau)}{\partial v}w_1 \\ &\frac{dw_1}{dt} = w_2 - 6\omega^{-2}\left[\frac{\partial^2\phi}{\partial v^2}w_1^{\ 2} + \frac{\partial\phi}{\partial v}w_2\right] \\ &\frac{dw_3}{dt} = \varphi'(u_o)v + f_o(v,\tau) + \frac{\partial f_1(v,\tau)}{\partial v}\phi(v,\tau) + M_2(v,\tau) - \varphi'(u_o)\,v \qquad (3.10) \end{aligned}\right)$$

The following lemma is therefore valid:

Lemma 3.2.for all $\mu \in (0,1)$, there are positive numbers w_1 and δ so that $w > w_1$ the system (3.10) has a unique $2\pi\omega^{-1}$ periodic solution v_ω in the ball $\|v(t) - u_o\| \le \delta.$ in addition holds equality $\lim\limits_{\omega\to\infty}\|u_\omega - u_o\|_{C_{\mu(R)}} = 0$ holds.

Here $C_{\mu(R)}$ is the ordinary Holder space of functions defined on the line $t \in R$ and ranging in R^n. This lemma is , in a sense , a corollary of theorem 2.1 in [10], Where abstract parabolic equations were considered. To provide a self-contained exposition, we outline the proof of lemma 3.2 (it differs from that in [10]).

By using the above change of variables, we reduce the equation (3.10) to the from:

$$\frac{dz}{dt} = pz + \chi(z,\tau) \qquad (3.11)$$

Where:

$$z = \begin{pmatrix} v \\ w_1 \\ w_2 \\ w_3 \end{pmatrix}_{4\times 1} \qquad p = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Phi'(u_o) & 0 & 0 & 0 \end{pmatrix}_{4\times 4}$$

And

$$\chi(z,\tau)=\begin{pmatrix} -4\omega^{-2}\dfrac{\partial\phi(v,\tau)}{\partial v}w_1 \\ -6\omega^{-2}\left[\dfrac{\partial^2\phi}{\partial v^2}w_1{}^2+\dfrac{\partial\phi}{\partial v}w_2\right] \\ \vdots \\ f_o(v,\tau)+\dfrac{\partial f_1(v,\tau)}{\partial v}\phi(v,\tau)+M_2(v,\tau)-\varphi'(u_o)v \end{pmatrix}_{4\times 1}$$

By virtue lemma (3.2) every T_ω -periodic solution z of equation (3.11) is representable in the from of:

$$z(t)=\left[E-e^{T_\omega P}\right]^{-1}\int_0^{T_\omega}e^{(T_\omega+t-s)p}\chi(z(s),\omega s)ds+\int_0^t e^{(t-s)P}\chi(z(s),\omega s)ds, \tag{3.12}$$

Where

$$T_\omega=\frac{2\pi}{\omega}\left[\frac{t_1}{2\pi}\omega\right].$$

Accordingly , with this equation we define operator as

$F:C_\mu\left([0,t_1]\times[0,\infty]\right)\to C_\mu[0,t_1]$

Putting

$$\left[F(z,\omega)\right](t)=z(t)-\left[E-e^{T_\omega P}\right]^{-1}\int_0^{T_\omega}e^{(T\omega+t-s)P}\chi(z(s),\omega s)ds-\int_0^t e^{(t-s)P}\chi(z(s),\omega s)ds,$$

$$\left[F(z,\infty)\right](t)=z(t)-\left[E-e^{T_\omega P}\right]^{-1}\int_0^{t_1}e^{(t_1+t-s)P}L(z(s))ds-\int_0^t e^{(t-s)P}L(z(s))ds. \tag{3.13}$$

for $\omega\in(0,\infty)$, where

$$L(z)=\begin{pmatrix}0\\0\\0\\\phi(v)\end{pmatrix}_{4\times 1}\qquad \varphi(v)=\langle f_o(v,\tau)+\frac{\partial f_1(v,\tau)}{\partial v}\phi(v,\tau)\rangle-\Phi'(u_o)\,v$$

$$\left[F(u_o,\infty)\right](t)=u_o-\left[E-e^{t_1P}\right]^{-1}\int_0^{t_1}e^{(t_1+t-s)P}L(u_o)ds-\int_0^t e^{(t-s)P}L(u_o)ds. \tag{3.14}$$

4.Main Result

In this section, containing basis prove lemma (3.1), we have use Theorem
Principle Operator in connection with operator $F(z,\omega)$ in the neighborhood point (u_o,∞). For that we establish the following facts:

I. $\left[F(u_o,\infty)\right](t)=0$ and $\left[(D_zF)(u_o,\infty)\right](t)\neq 0$

We omit the proof I.

II. The mapping $F(z,\omega)$ is continuous over the interval (u_o,∞) and in this neighborhood has a partial derivative $(D_z F)(z,\omega)$, also continuous over (u_o,∞). We have studied the statement (lemma 3.2) and attempted to prove that as $F(z,\omega)$ as continuous over (u_o,∞)

For that we established that $\forall\varepsilon>0,\exists\delta:\|z(t)-u_o\|_{C_\mu}<\delta,\omega>\frac{1}{\delta}$ is valid in the following estimation :

$$\|F(z,\omega)-F(u_o,\infty)\|_{C_\mu}\le\varepsilon. \tag{4.1}$$

We also noted that, $F(z,\omega)=z+\bar{F}(z,\omega)$ and $F(u_o,\infty)=u_o+\bar{F}(u_o,\infty)$, with regard $\delta_1>\varepsilon/2$. to prove (4.1) it is sufficient to show

$$\|\bar{F}(z,\omega)-\bar{F}(u_o,\infty)\|_{C_\mu}\le\varepsilon/2 \tag{4.2}$$

Using this inequality

$$\|\bar{F}(z,\omega)-\bar{F}(u_o,\omega)\|_{C_\mu}\le\|\bar{F}(z,\omega)-\bar{F}(u_o,\omega)\|_{C_\mu}+\|\bar{F}(u_o,\omega)-\bar{F}(u_o,\omega)\|_{C_\mu} \tag{4.3}$$

We prove the validity of the inequality,

$$\|\bar{F}(z,\omega)-\bar{F}(u_o,\omega)\|_{C_\mu}\le\varepsilon/4 \tag{4.4}$$

By virtue representation :

$$\left[\bar{F}(z,\omega)\right](t)=-\left[E-e^{T_\omega P}\right]^{-1}\int_0^{T_\omega}e^{(T_\omega+t-s)P}\chi(z(s),\omega s)ds-\int_0^t e^{(t-s)P}\chi(z(s),\omega s)ds$$

And

$$\left[\bar{F}(u_0,\omega)\right](t)=-\left[E-e^{T_\omega P}\right]^{-1}\int_0^{T_\omega}e^{(T_\omega+t-s)P}\chi(u_o,\omega s)ds-\int_0^t e^{(t-s)P}\chi(u_o,\omega s)ds. \tag{4.5}$$

Having

$$\begin{aligned}\|\bar{\mathrm{F}}(z,\omega)-\bar{\mathrm{F}}(u_0,\omega)\|_{C_{([0,t_1])}}&=\sup_{0\le s\le t_1}\Bigg|-\left[\mathrm{E}-e^{T_\omega P}\right]^{-1}\int_0^{T_\omega}e^{(T_\omega+t-s)P}\left[\chi(z(s),\omega s)-\right.\\&\left.-\chi(u_0,\omega s)\right]ds-\int_0^t e^{(t-s)P}\left[\chi(z(s),\omega s)-\chi(u_0,\omega s)\right]ds\le\\&\le C_1\sup_{0\le s\le t_1}\left|\chi(z(s),\omega s)-\chi(u_0,\omega s)\right|+C_2\sup_{0\le s\le t_1}\left|\chi(z(s),\omega s)-\right.\\&\left.-\chi(u_0,\omega s)\right|\le C_1\sup_{0\le s\le t_1}\left|\int_0^1\frac{d\chi}{dz}(z+\theta(u_0-z),\omega s)d\theta(z(s)-u_0)\right|+\end{aligned}$$

$$+C_2 \sup_{0\le s\le t_1} \int_0^1 \frac{d\chi}{dz}\left(z+\theta(u_0-z),\omega s\right)\, d\theta\left(z(s)-u_0\right)\le$$

$$\le C_3\left\|z(t)-u_0\right\|_{C_{([0,t_1])}} + C_4\left\|z(t)-u_0\right\|_{C_{([0,t_1])}} \le C_3\delta_1 + C_4\delta_1 \equiv C_5\delta_1, \tag{4.6}$$

Where C_1, C_2, C_3, and, C_4 constant

The required estimate is:

$$\left\|\bar{F}(z,w)-\bar{F}(u_0,\infty)\right\|_{C^1_{([0,t_1])}} = \left\|\bar{F}(z,\omega)-\bar{F}(u_0,\omega)\right\|_{C_{([0,t_1])}} + \frac{d}{dt}\left\|\bar{F}(z,\omega)(t)-\bar{F}(u_0,\omega)(t)\right\|_{C_{([0,t_1])}} \tag{4.7}$$

If we use representation:

$$\frac{d}{dt}\left[\left[\bar{F}(z,\omega)\right](t)\right] = -\left[E-e^{T_\omega P}\right]^{-1}\int_0^{T_\omega} Pe^{(T_\omega+t-s)P}\chi\left(z(s),\omega s\right)ds - \\ -\int_0^t e^{(t-s)P}\chi\left(z(s),\omega s\right)ds-\chi\left(z(t),\omega t\right) \tag{4.8}$$

$$\frac{d}{dt}\left[\bar{F}(u_0,\omega)\right](t) = -\left[E-e^{T_\omega P}\right]^{-1}\int_0^{T_\omega} e^{(T_\omega+t-s)P}X\left(u_0,\omega s\right)ds - \\ -\int_0^t Pe^{(t-s)P}X\left(u_0,\omega s\right)ds - X\left(u_0,\omega t\right). \tag{4.9}$$

We have:

$$\left\|\frac{d}{dt}\left\{\left[\bar{F}(z,\omega)\right](t)-\left[\bar{F}(u_0,\omega)\right](t)\right\}\right\|_{C_{([o,t_1])}} =$$

$$= \sup_{0\le t\le t_1}\left|\left[E-e^{T_\omega P}\right]^{-1}\int_0^{T_\omega} e^{(T_\omega+t-s)P}\left[\chi\left(z(s),\omega s\right)-\chi\left(u_0,\omega s\right)\right]ds -\right.$$

$$-\int_0^t Pe^{(t-s)P}\left[\chi\left(z(s),\omega s\right)-\chi\left(u_0,\omega s\right)\right]ds - \left[\chi\left(z(t),\omega t\right)-\chi\left(u_0,\omega t\right)\right]\le$$

$$\le b_1 \sup_{0\le s\le t_1} \left|\chi(z(s),\omega s)-\chi(u_0,\omega s)\right| + \sup_{0\le s\le t_1}\left[\chi(z(t),\omega t)-\chi(u_0,\omega t)\right]+$$

$$+b \sup_{0\le s\le t_1} \left|\chi(z(s),\omega s)-\chi(u_0,\omega s)\right| \le$$

$$\le b_1 \sup_{0\le s\le t_1} \int_0^1 \frac{\partial\chi}{\partial z}\left(z+\theta(u_0-z),\omega s\right)\, d\theta\left(z(s)-u_0\right)\Big| +$$

$$+b \sup_{0\le s\le t_1} \left|\int_0^1 \frac{\partial\chi}{\partial z}\left(z+\theta(u_0-z),\omega s\right) d\theta\left(z(s)-u_0\right)\right| +$$

$$+ \sup_{0\le s\le t_1} \left|\int_0^1 \frac{\partial\chi}{\partial z}\left(z+\theta(u_0-z),\omega t\right) d\theta\left(z(t)-u_0\right)\right| \le$$

$$\le b_2 \left\|z(t)-u_0\right\| C_{([0,t_1])} + b_3 \left\|z(t)-u_0\right\| C_{([0,t_1])} \le b_2\delta_1 + b_3\delta_1 \equiv b_4\delta_1. \tag{4.10}$$

If we apply estimators (4.6),(4.10) and known interpolation formula:

$$\frac{\left|u(t_2)-u(t_1)\right|}{\left|t_2-t_1\right|^{\mu}} \le \|u\|^{\mu}_{C^1}\left(2\|u\|\right)^{1-\mu},$$

We derive $C_5\delta_1+\left(2C_5\delta_1\right)^{1-\mu}\left(b_4\delta_1\right)^{\mu} < \varepsilon/4$. For δ_1, we had the estimator (4.4). now we prove there exists δ_2, $0<\delta_2\le\delta_1$, with $\omega > \frac{1}{\delta_2}$. The valid inequality is now

$$\left\|\bar F(u_0,\omega)-\bar F(u_0,\infty)\right\|_{C_\mu} < \varepsilon/4. \tag{4.11}$$

By virtue (3.11) and (4.5) we have:

$$\bar F(u_o,\omega)-\bar F(u_o,\infty) = \left[E-e^{t_1P}\right]^{-1}\int_0^{t_1} e^{(t_1+t-s)P}\left[\chi(u_0,\omega s)-L(u_0)\right]ds +$$

$$+\int_0^t e^{(t-s)P}\left[\chi(u_0,\omega s)-L(u_0)\right]ds. \tag{4.12}$$

We define Δ_1 to be:

$$\Delta_1 = \int_0^{t_1} e^{(t_1+t-s)P}\phi_1(\omega s)ds, \text{ where } \phi_1(\omega s) = \left[E-e^{t_1P}\right]^{-1}\left[\chi(u_0,\omega s)-L(u_0)\right]$$

At that point, we can write it in the from of:

$$\Delta_1 = \sum_{i=0}^{n} \int_{S_i}^{S_{i+1}} \left[e^{(t_1+t-s)P} \phi_1(\omega s) - e^{(t_1+t-s_i)P} \phi_1(\omega s) \right] ds +$$

$$+\sum_{i=0}^{n} \int_{S_i}^{S_{i+1}} e^{(t_1+t-s)P} \phi_1(\omega s) ds + \int_{S_{n+1}}^{t_1} e^{(t_1+t-s)P} \phi_1(\omega s) ds \equiv$$

$$\equiv \Delta_{11} + \Delta_{12} + \Delta_{13}, \qquad (4.13)$$

here $S_i = \frac{2\pi i}{\omega}$, $i = 1,2,3,\ldots,n$ and $|t_1 - S_{n+1}| < \frac{2\pi}{\omega}$. We estimate that:

$$\|\Delta_{11}\|_C = \sup_{0\le s\le t_1} \left| \sum_{i=0}^{n} \int_{S_i}^{S_{i+1}} \left[e^{(t_1+t-s)P} \phi_1(\omega s) - e^{(t_1+t-s_i)P} \phi_1(\omega s) \right] ds \right| \le$$

$$\le n\frac{2\pi}{\omega} c \frac{2\pi}{\omega} \le \frac{ct_1 2\pi}{\omega} < c_1\delta_2. \qquad (4.14)$$

$$\|\Delta_{12}\|_C = \sup_{0\le s\le t_1} \left| \sum_{i=0}^{n} \int_{S_i}^{S_{i+1}} e^{(t_1+t-s_i)P} \phi_1(\omega s) ds \right| = 0$$

And,

$$\|\Delta_{13}\|_C = \sup_{0\le s\le t_1} \left| \sum_{i=0}^{n} \int_{S_{n+1}}^{t_1} e^{(t_1+t-s)P} \phi_1(\omega s) ds \right| \le c_2 t_1 \|\phi_1\|_C \le c_2\delta_2.$$

$$(4.15)$$

If we substitute relation (4.14) and (4.15) in the equation (4.13), we have:

$$\|\Delta_1\|_C < a_1\delta_2 \qquad (4.16)$$

Where $C_1\delta_2 + C_2\delta_2 \equiv a_1\delta_2$ and $\mathbf{C_1}, \mathbf{C_2}$ – are constants. Similarly, estimation of the second term in first part of equation (4.12), we have:

$$\left\|\bar{F}(u_0,\omega) - \bar{F}(u_0,\infty)\right\|_{C_\mu} \le a_1\delta_2 \qquad (4.17)$$

a-constant.

Now, we estimate that:

$$\left\|\bar{F}(u_0,\omega) - \bar{F}(u_0,\infty)\right\|_{C^1([0,t_1])} = \left\|\bar{F}(u_0,\omega) - \bar{F}(u_0,\infty)\right\|_{C([0,t_1])} +$$

$$+\left\|\frac{d}{dt}\left\{\left[\bar{F}(u_0,\omega)\right](t) + \left[\bar{F}(u_0,\infty)\right](t)\right\}\right\|_C. \qquad (4.18)$$

By virtue of (3.14) and (4.5), the following is significant:

$$\frac{d}{dt}\left[\bar{F}\left[(u_0,\omega)\right](t)-\left[\bar{F}(u_0,\infty)\right](t)\right]=$$

$$\left[E-e^{t_1P}\right]^{-1}\int_0^{t_1}Pe^{(t_1+t-s)P}\left[\chi(u_0,\omega s)-L(u_0)\right]ds-[\chi(u_0,\omega t)-L(u_0)]-$$

$$-\int_0^{t_1}e^{(t_1+t-s)}[\chi(u_0,\omega s)-L(u_0)]ds.$$

We propose the following notation:

$$\Delta_1'=\int_0^{t_1}e^{(t_1+t-s)P}\Psi_2(\omega s)ds \text{ and } \Delta_1''=\int_0^{t_1}e^{(t_1+t-s)P}\Psi_2(\omega s)ds,$$

Where $\Psi_2(\omega s)=\left[E-e^{t_1P}\right]^{-1}P\left[\chi(u_0,\omega s)-L(u_0)\right]$ and $\tilde{\Psi}_2(\omega s)=P\left[\chi(u_0,\omega s)-L(u_0)\right]$.

So that similarly inequality (4.16) has: $\left\|\Delta_1'\right\|_C<a_3\delta_2$, $\left\|\Delta_1''\right\|_C<a_4\delta_2$ and that

$$\sup_{0\le s\le t_1}\left|\chi(u_0,\omega t)-L(u_0)\right|<a_5, \tag{4.19}$$

Where a_3, a_4, and a_5 – constant. From equation (4.17) and (4.19) having:

$$\left\|\bar{F}(u_0,\omega)-\bar{F}(u_0,\infty)\right\|_{C^1}\le a_6 \tag{4.20}$$

Where $a_6\equiv a_2\delta_2+a_3\delta_2+a_4\delta_2+a_5$. Select $\delta_2<\delta_1$, we selected so that satisfying relation is

$$\left\|\bar{F}(u_0,\omega)-\bar{F}(u_0,\infty)\right\|_{C_\mu}<\varepsilon/4. \tag{4.21}$$

If we substitute expression (4.4) and (4.21) in inequality (4.3), we have illustrated estimation (4.2).

It is necessary to prove that $(D_zF)(z,\omega)$ is continuous at the point (u_0,∞) for that , we prove that $\forall\varepsilon>0,\exists\vartheta_3:\ \delta_3$

$\left\|z(t)-u_0\right\|_{C_\mu}<\delta_3,\quad |\omega|>1/\delta_3,\quad \forall y:\|y\|_{C_\mu}=1$ and the estimate is,

$$\left\|(D_zF)(z,\omega)y-(D_zF)(u_0,\infty)\right\|_{C_\mu}<\varepsilon, \tag{4.22}$$

Here

$$\left[D_zF(z,\omega)\right]y(t)=\left[E-e^{T_\omega P}\right]^{-1}\int_0^{T_\omega}e^{(T_\omega+t-s)p}\frac{\partial\chi(z(s),\omega s)}{\partial z}y(s)ds$$
$$-\int_0^{t}e^{(t-s)p}\frac{\partial\chi(z(s),\omega s)}{\partial z}y(s)ds \tag{4.23}$$

Apply the inequality:

$$\left\|(D_zF)(z,\omega)y-(D_zF)(u_0,\infty)y\right\|_{C_\mu}\le\left\|(D_zF)(z,\omega)y-(D_zF)(u_0,\omega)y\right\|_{C_\mu}+$$

$$+\left\|(D_zF)(z,\omega)y-(D_zF)(u_0,\infty)y\right\|_{C_\mu}. \qquad (4.24)$$

To prove that:

$$\left\|(D_zF)(z,\omega)y-(D_zF)(u_0,\omega)y\right\|_{C_\mu}<\varepsilon/2. \qquad (4.25)$$

We estimate the value as:

$$\left\|(D_zF)(z,\omega)y-(D_zF)(u_0,\omega)y\right\|_{C}$$

And

$$\left\|(D_zF)(z,\omega)y-(D_zF)(u_0,\omega)y\right\|_{C^1} \qquad (4.26)$$

Also

$$\left\|(D_zF)(z,\omega)y-(D_zF)(u_0,\omega)y\right\|_{C}=$$

$$\sup_{0\le t\le t_1}\left\|\left[E-e^{T_\omega P}\right]^{-1}\int_0^{T_\omega}e^{(T_\omega+t-s)P}\left[\frac{\partial\chi(z(s),\omega s)}{\partial z}-\frac{\partial\chi(u_0,\omega s)}{\partial z}\right]y(s)\,ds-\right.$$

$$\left.-\int_0^t e^{(t-s)P}\left[\frac{\partial\chi(z(s),\omega s)}{\partial z}-\frac{\partial\chi(u_0,\omega s)}{\partial z}\right]y(s)\,ds\le\right.$$

$$\le C_1\sup_{0\le s\le t_1}|y(s)|\sup_{0\le s\le t_1}\left[\frac{\partial\chi(z(s),\omega s)}{\partial z}-\frac{\partial\chi(u_0,\omega s)}{\partial z}\right]+$$

$$+C_2\sup_{0\le s\le t_1}|y(s)|\sup_{0\le s\le t_1}\left[\frac{\partial\chi(z(s),\omega s)}{\partial z}-\frac{\partial\chi(u_0,\omega s)}{\partial z}\right]\le$$

$$\le C_1\sup_{0\le s\le t_1}\left|\int_0^1\frac{\partial^2\chi}{\partial z^3}\left(z+\theta(u_0-z),\omega s\right)d\theta\left(z(s)-u_0\right)\right|+$$

$$+C_2\sup_{0\le s\le t_1}\left|\int_0^1\frac{\partial^2\chi}{\partial z^2}\left(z+\theta(u_0-z),\omega t\right)d\theta\left(z(t)-u_0\right)\right|\le$$

$$\le C_3\sup_{0\le s\le t_1}|z(s)-u_0|+C_4\sup_{0\le s\le t_1}|z(s)-u_0|\le$$

$$\le C_3\delta_3+C_4\delta_3\equiv C_5\delta_3, \qquad (4.27)$$

Where $C_1, C_2, C_3\,C_4$ and C_5 as constants are also $0\le\theta\le1$. Now we consider

$$\left\|(D_zF)(z,\omega)y-(D_zF)(u_0,\infty)y\right\|_{C^1}=\left\|(D_zF)(z,\omega)y-(D_zF)(u_0,\omega)y\right\|_{C}$$

$$+\left\|\frac{d}{dt}\left[(D_zF)(z,\omega)y\right](t)-\frac{d}{dt}\left[(D_zF)(u_0,\infty)y\right](t)\right\|_{C}. \qquad (4.28)$$

We estimate the following expression

$$\left\| \frac{d}{dt}\left[(D_z F)(z,\omega) y \right](t) - \frac{d}{dt}\left[(D_z F)(u_0,\infty) y \right](t) \right\|_C =$$

$$= \sup_{0\le t\le t_1} \left| \left[E - e^{T_\omega P} \right]^{-1} \int_0^{T_\omega} e^{(T_\omega + t - s)P} \left[\frac{\partial \chi(z(s),\omega s)}{\partial z} - \frac{\partial \chi(u_0,\omega s)}{\partial z} \right] y(s)\, ds + \right.$$

$$\left. + \left[\frac{\partial \chi(z(t),\omega t)}{\partial z} - \frac{\partial \chi(u_0,\omega t)}{\partial z} \right] y(t) \right| \le b\delta_3, \qquad (4.29)$$

Where b-constant

From estimation (4.27) and (4.29) , having applied interpolation formulas, we arrive at:

$$C_5\delta_3 + (2C_5\delta_3)^{1-\mu}(b\delta_3)^{\mu} < \varepsilon/2.$$

This proves estimation (4.25). Now we prove that existence $0<\delta_4\le\delta_3$, so that with

$$|\omega| > 1/\delta_4 ,$$

$$\left\| (D_z F)(u_0,\omega) y - (D_z F)(u_0,\infty) y \right\|_{C_\mu} < \varepsilon/2 \qquad (4.30)$$

In light of (4.23), we derive:

$$\left[(D_z F)(u_0,\infty) \right] y(t) = \left[E - e^{T_\omega P} \right]^{-1} \int_0^{T_\omega} e^{(T_\omega + t - s)P} \frac{\partial \chi(u_0,\omega s)}{\partial z} y(s)\, ds +$$

$$+ \int_0^t e^{(t-s)P} \frac{\partial \chi(u_0,\omega s)}{\partial z} y(s)\, ds. \qquad (4.31)$$

$$\left[(D_z F)(u_0,\infty) \right] y(t) = \left[E - e^{t_1 P} \right]^{-1} \int_0^{t_1} e^{(t_1 + t - s)P} \frac{\partial L(u_0)}{\partial z} y(s)\, ds +$$

$$+ \int_0^t e^{(t-s)P} \frac{\partial L(u_0)}{\partial z} y(s)\, ds, \qquad (4.32)$$

and having:

$$\left\| (D_z F)(u_0,\omega) y - (D_z F)(u_0,\infty) y \right\|_{C_\mu} =$$

$$= \sup_{0\le t\le t_1} \left| \left[E - e^{t_1 P} \right]^{-1} \int_0^{t_1} e^{(t_1 + t - s)P} \left[\frac{\partial \chi(u_0,\omega s)}{\partial z} - \frac{\partial L(u_0)}{\partial z} \right] y(s)\, ds + \right.$$

$$\left. + \int_0^t e^{(t-s)P} \left[\frac{\partial \chi(u_0,\omega s)}{\partial z} - \frac{\partial L(u_0)}{\partial z} \right] y(s)\, ds \right|. \qquad (4.33)$$

We now introduce notation: $\Delta_3 = \int_0^{t_1} e^{(t_1 + t - s)P} \Psi_3(\omega s)\, y(s)\, ds,$

Where $\Psi_3(\omega s)=\left[E-e^{t_1P}\right]^{-1}\left[\frac{\partial\chi(u_0,\omega s)}{\partial z}-\frac{\partial L(u_0)}{\partial z}\right]$. It is how possible to write Δ_3 in the from:

Where

$$\Delta_3=\sum_{i=0}^{n}\int_{S_i}^{S_{i+1}}\left[e^{(t_1+t-s)P}\Psi_3(\omega s)y(s)-e^{(t_1+t-s_i)P}\Psi_3(\omega s)y(s_i)\right]ds+$$

$$+\sum_{i=0}^{n}\int_{S_i}^{S_{i+1}}e^{(t_1+t-s)P}\Psi_3(\omega s)y(s_i)ds+$$

$$+\int_{S_{n+1}}^{t_1}e^{(t_1+t-s)P}\Psi_3(\omega s)y(s_i)ds\equiv\Delta_{31}+\Delta_{32}+\Delta_{33},$$

Here $S_i=\frac{2\pi i}{\omega}$, $i=1,2,3,\cdots,n,and$ $\left|t_1-S_{n+1}\right|<\frac{2\pi}{\omega}$.

Now we estimate:

$$\left\|\Delta_{31}\right\|_{C([0,t_1])}=\sup_{0\le s\le t_1}\left|\sum_{i=0}^{n}\int_{S_i}^{S_{i+1}}\left[e^{(t_1+t-s)P}y(s)-e^{(t_1+t-s_i)P}y(s_i)\right]\Psi_3(\omega s)ds\right|$$

$$\le n\frac{2\pi}{\omega}d\frac{2\pi}{\omega}\le\frac{dt_1 2\pi}{\omega}<d_1\delta_4, \qquad (4.34)$$

Where d_1,d_2 -are constant.

$$\left\|\Delta_{32}\right\|_{C([0,t_1])}=\sup_{0\le s\le t_1}\left|\sum_{i=0}^{n}\int_{S_i}^{S_{i+1}}e^{(t_1+t-si)P}\Psi_3(\omega s)y(s_i)ds\right|\equiv 0$$

$$(4.35)$$

$$\left\|\Delta_{33}\right\|_{C([0,t_1])}=\sup_{0\le s\le t_1}\left|\int_{S_{n+1}}^{t_1}e^{(t_1+t-s)P}\Psi_3(\omega s)y(s)ds\right|\le d_2t_1\left\|\Psi_3\right\|_{C([0,t_1])}<$$

$$<d_2\delta_4. \qquad (4.36)$$

If we substitute relation (4.34) and (4.36) in equation (4.33) we arrive at

$$\left\|\Psi_3\right\|_{C([0,t_1])}<d_3\delta_4$$

Where $d_1\delta_4+d_2\delta_4\equiv d_3\delta_4, d_2$ -constant. (4.37)

Similarly, the estimation of the second terms in the first part equation (4.32) leads to

$$\left\|(D_z F)(u_0,\omega)y-(D_z F)(u_0,\infty)y\right\|_{C_{([0,t_1])}} < d_4\delta_4, \qquad (4.38)$$

d_4-constant

Now , we estimate

$$\left\|(D_z F)(u_0,\omega)y-(D_z F)(u_0,\infty)y\right\|_{C^1} \le \left\|(D_z F)(u_0,\omega)y-(D_z F)(u_0,\omega)y\right\|_{C}+$$
$$+\left\|\frac{d}{dt}\left[(D_z F)(u_0,\omega)y\right](t)-\frac{d}{dt}\left[(D_z F)(u_0,\infty)y\right](t)\right\|_{C} \qquad (4.39)$$

With (4.32) and (4.33):

$$\frac{d}{dt}\left[(D_z F)(u_0,\omega)y\right](t)-\frac{d}{dt}\left[(D_z F)(u_0,\infty)y\right](t)=$$
$$=\left[E-e^{t_1P}\right]^{-1}\int_0^{t_1} Pe^{(t_1+t-s)P}\left[\frac{\partial\chi(u_0,\omega s)}{\partial z}-\frac{\partial L(u_0)}{\partial z}\right]y(s)\,ds+$$
$$+\left[\frac{\partial\phi(u_0,\omega s)}{\partial z}-\frac{\partial L(u_0)}{\partial z}\right]y(t).$$

We define by Δ_4 the expression: $\Delta_4=\int_0^{t_1} e^{(t_1+t-s)P}\Psi_4(\omega s)\,y(s)\,ds,$

Where $\Psi_4(\omega s)=\left[E-e^{t_1P}\right]^{-1}P\left[\frac{\partial\chi(u_0,\omega s)}{\partial z}-\frac{\partial L(u_0)}{\partial z}\right].$

Similarly, (4.16) and (4.19) give the following estimate

$\left\|\Delta_4\right\|_{C_{([0,t_1])}} < d_5\delta_4$, where d_5-constant. $\qquad (4.40)$

And thus:

$$\sup_{0\le t\le t_1}\left|\left[\frac{\partial\chi(u_0,\omega s)}{\partial z}-\frac{\partial L(u_0)}{\partial z}\right]y(t)\right| \le d_6\|y\|_C < d_6, \qquad (4.41)$$

Where d_6 -constant .

From estimation (4.38),(4.40) and (4.41):

$$\left\|(D_z F)(u_0,\omega)y-(D_z F)(u_0,\infty)y\right\|_{C^1_{([0,t_1])}} < d_7$$

Where $d_7 \equiv d_4\delta_4+d_5\delta_4+d_6$. The number , selected so that the $\delta_4<\delta_3$, has fulfilled relation (4.30) with regard to expression (4.25) and (4.30) and validates estimate (4.22). thus,(the statement I) is proved as well as this lemma 3.1.This concludes the proof of lemma 3.1.

The proof theorem (2.1) implies lemma 3.1; the system equation (3.10) has a unique $2\pi\omega^{-1}$-periodic solution u_ω. with 2- the change variable and using Implicit Operator theorem, we define operator as

$$\Psi : C_\mu\left(\left[0,t_1\right]\right)\times C_\mu\left(\left[0,t_1\right]\right)\times\left([1,\infty]\right)\rightarrow C_\mu\left(\left[0,t_1\right]\right)$$

Here

$\left[\Psi(u,v,\omega)\right](t)=u(t)-v(t)-\omega^{-2}\phi\left(v(t),\omega t\right)$ and $\left[\Psi(u,v,\infty)\right](t)=u(t)-v(t)$.

It is therefore not difficult to see that $\left[\Psi(u_0,u_0,\infty)\right](t)=0$ and $D_v\Psi(u_0,u_0,\infty)(t)=-1$, by Implicit Operator Theorem, there exist positive number ω_2,ρ_1 and $\rho_2=\min\left(\omega_2,\rho_1\right)$ with $\omega>\omega_2$ and $\|v-u_0\|_{C_\mu([0,t_1])}\le\rho_1$ the equation (3.2) has the solution $u_{(v,\omega)}$ in ball $\|u-u_0\|_{C_\mu\left(\left[0,t_1\right]\right)}<\rho_2$ in addition $\lim\limits_{\omega_2\to\infty}\rho_2=0$, $u_{(v,\omega)}$ is $2\pi\omega\left[{t_1\omega}/{2\pi}\right]$-periodic and containable on the axis $t\in R$ is a $2\pi\omega^{-1}$-periodic solution u_ω. From this , we have the following:

$$\lim_{\omega\to\infty}\left\|u_\omega-u_o\right\|_{C_\mu(R)}=0$$

The proof of theorem (2.1) is therefore complete.

References

[1] V.V. Sazonov, Periodic Solutions of set of Ordinary Differential Equations with a Large Parameter , Journal of Applied Mathematics and Mechanics, vol. 47, Issue 5, pp. 579-588, 2003.
[2] Zhongjin Guo, A.Y.T. Leng, Y.C. Liu, and H.X. Yang, Bifurcating Periodic Solution for a Class of First-Order Nonlinear Delay Differential Equation after Hop Bifurcation, Applied Mathematical Modeling, in press, 2011.
[3] V.B. Levenshtam, Asymptotic Expansions of Periodic Solutions of Ordinary Differential Equations with Large High-frequency Terms, Mathematics and Statistics Differential Equations, vol. 44, no.1, pp. 54-70, 2005.
[4] P.J. Mckenna and K.S. Moore, Multiple Periodic Solutions to a Suspension Bridge Ordinary Differential Equation, Nonlinear Differential Equations, vol. J. Diff. Eqns, PP. 183-199, Conf. 05. 2000.
[5] Jose Luis Bravo and Pedro J. Torres, Periodic Solutions of a Singular Equation with Indefinite Weight, Advanced Nonlinear, vol. 10, pp. 927-938, 2010.
[6] Alexander Gutierrez and Pedro J. Torres, Periodic Solutions of Lienard Second Order Differential Equations with One or Two Weak Singularities, Differential Equations&Applications, vol. 3. no. 3, pp. 375-384, 2011.
[7] Haiyan Wang, Positive Periodic Solutions of Singular Systems of First Order Ordinary Differential Equations, Applied Mathematics and Computation, in press, 2011.
[8] Huafeng Xiao, Jianshe Yu and Zhiming Guo, The Clark Dual and Multiple Periodic Solutions of Delay Differential Equations, Boundary Value Problems, vol.44. 2011.
[9] N.M. Krylov and N.N. Bogolyubov, Introduction to Nonlinear Mechanics, Kiev. Acad. Sci, USSR, 1937.
[10] N.N. Bogoliubov and Yu.A. Mitropolsky, Asymptotic Method of the Theory of Nonlinear Oscillations, Moscow Science, USSR, 1974.
[11] N.N. Bogoliubov," Perturbation Theory in Nonlinear Mechanics," *Sat.* Institution Building Mechanics, USSR, vol. 4, pp. 9-34, 1950.

[12] P.L. Kapitza, “Dynamic Stability of Private Enterprise with a Vibrating Point of Suspension,” *PMM*., vol. 59, no. 6, pp.922-929, 1995.
[13] V.N. Chelomei, “On the Possibility of Increasing the Stability of Elastic Systems by Means of Vibration,” *DAN. USSR*, vol. 110, no. 3, pp. 345-347,1956.
[14] V.M. Volosov, “Higher Approximations in the Averaging”, DAH. USSR, vol.137, no. 5, pp. 1022-1025, 1961.
[15] V.V. Strygin, “Modification of the Averaging Method for Finding Higher Approximations,” *PMM*. vol. 48, no. 6, pp.1042-1045, 1984.
[16] V.E. Yudovich, "Vibration Dynamics of Systems with Constraints." Dep. In VINITII, no. 1408-B2003, Moscow, Russian, 7. 07. 2003.
[17] H.J. Abood, "Asymptotic Integration Problem of Periodic Solutions of Ordinary Differential Equation Containing a Large High-Frequency Terms," Dep. In VINITI, no. 338-B2004, 28 pages, Moscow, Russian, 26. 02. 2004.
[18] H.J. Abood," Asymptotic Integration Problem of Periodic Solutions of Ordinary Differential Equation of Third Order with Rapidly Oscillating Terms, " Dep. In VINITI, no. 1357-B2004, 32 pages, Moscow, Russian, 05. 08. 2004.
[19]V.B. Levenshtan, "Asymptotic Analysis of Some Classes of Ordinary Differential Equations with Large High-Frequency Terms, " Journal of Mathematical Sciences, vol. 63,no.1, pp. 89-110, 2009.
[20] V.B. Levenshtam, "Kantorovich Theorem and Justification of Asymptotic of Solutions of Differential Equations with Large High-Frequency Summands," Journal of Mathematical Sciences, vol. 163, no. 2, pp. 228-234, 2009.