Casting Retrospective Glances of David Hilbert in the History of Pure Mathematics

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Abstract

This study was conducted with the view of reconfirming the historical contributions of David Hilbert in the form of a review in the area of mathematics. It emerged that he contributed to the growth of especially the Nullstellensatz and was the first person to prove it. The implications and conclusions are that the current trend in the said area of mathematics could be appreciated owing to contributions and motivations of such a mathematician.

Keywords: David Hilbert, Nullstellensatz, affine variety, natural number, general contribution

1.0 Introduction

The term history came from the Greek word ἱστορία - historia, which means "inquiry, knowledge acquired by investigation"(Santayana, …….) and it is the study of the human past. History can also mean the period of time after writing was invented. David Hilbert was a German mathematician who was born on 23rd January, 1862 and his demise occurred on 14th February, 1943. His parents were Otto and Maria Therese. He was born in the Province of Prussia, Wahlau and was the first of two children and only son. As far as the 19th and early 20th centuries are concerned he is deemed as one of the most influential and universal mathematicians. He reduced geometry to a series of axioms and contributed substantially to the establishment of the formalistic foundations of mathematics. His work in 1909 on integral equations led to 20th-century research in functional analysis. Hilbert extensively modified the mathematics of invariants—the entities that are not altered during such geometric changes as rotation, dilation, and reflection. Hilbert proved the theorem of invariants—that all invariants can be expressed in terms of a finite number. He also proved the conjecture in number theory that for any n, all positive integers are sums of a certain fixed number of nth powers; for example, $5 = 2^2 + 1^2$, in which $n = 2$. [Van der Waerden 1985]. Invariant theory and axiomatization of geometry were some of the fundamental ideas he came out with. In functional analysis too he formulated the Hilbert spaces theory.

2.0 The Life of David Hilbert

In modern mathematics Hilbert and his students made giant strides in establishing rigor in modern mathematical physics and is among founders of the proof theory and mathematical logic.

The following were students of Hilbert: Hermann Weyl, Emanuel Lasker, Ernst Zermelo, and Carl Gustav Hempel. His assistant was John von Neumann . He had 69 Ph.D. students Otto Blumenthal , Felix Bernstein , Hermann Weyl , Richard Courant , Erich Hecke , Hugo Steinhaus, Wilhelm Ackermann.

His work on invariant functions gave rise to his famous finiteness theorem. Meanwhile Paul Gordon had twenty years earlier illustrated the theorem of the finiteness of generators for binary forms using a complex computational approach. As a sequel to this Hilbert depicted the Hilbert's basis theorem. Around 1909, Hilbert dedicated himself to the study of differential and integral equations; his work had direct consequences for important parts of modern functional analysis. In order to carry out these studies, Hilbert introduced the concept of an infinite dimensional Euclidean space, later called Hilbert space. Hilbert spaces are an important class of objects in the area of functional analysis, particularly of the spectral theory of self-adjoint linear operators that grew up around it during the 20th century. [Van der Waerden 1985]. When Hilbert wanted to publish this paper the comment for rejection was that

> This is not Mathematics. This is Theology.

Klein made it to be published without any adjustment. Hilbert sent a second paper and this was the comment from Klein

> Without doubt this is the most important work on general algebra that the Annalen has ever published.

Gordon in a later discourse nuncupated that:

> I have convinced myself that even theology has its merits.

Hilbert chalked lots of successes even though not without problems. He published

Foundations of Geometry in 1899 which proposed a formal set, the Hilbert's axioms, which substitued the traditional axioms of Euclid.

Hilbert disclosed that elements such as point, line, plane, and a host of others could be substituted by tables, chairs, glasses of beer and other such objects. His axiom unify both the plane geometry and solid geometry of Euclid in a single system. Hilbert's scientific activity can be roughly divided into six periods, according to the years of publication of the results: up to 1893 (at Königsberg), algebraic forms; 1894–1899, algebraic number theory; 1899–1903, foundations of geometry; 1904–1909, analysis (Dirichlet's principle, calculus of variations, integral equations, Waring's problem); 1912–1914, theoretical physics; after 1918, foundations of mathematics. [Derbyshire 2004].

3.0 Achievements of Hilbert

At the International Congress of Mathematicians in Paris in 1900, Hilbert presented a formidable list of 23 unsolved problems. Hilbert starts his treatment of plane geometry with five undefined concepts: point, line, on (a relation holding between a point and a line), between (a relation between a point and a pair of points), congruent (a relation between a pair of points). He then lists fifteen axioms from which he develops all of plane geometry. His treatment of solid geometry is based on twenty-one axioms involving six undefined concepts. [Apostol. T. M. 1997]

Part of Hilbert's sixteenth problem asked for the maximum number of limit cycles of the system $\{x' = A(x,y),\ y' = B(x,y)\}$ where A and B are polynomials. If A and B are polynomials of degree n, then the maximum number is known as the Hilbert Number or the Hilbert Function, H_n. It is known that $H_0 = 0,\ H_1 = 0,\ H_2 \geq 4,\ H_3 \geq 5$ and $[(n-1)/2] \leq H_n$ if n is odd. The Hilbert transform is deifned as $\upsilon(\xi) = \int_{-\infty}^{\infty} \frac{1}{\pi(x-\xi)} u(x)\,dx,\ u(x) = \int_{-\infty}^{\infty} \frac{1}{\pi(\xi - x)} \upsilon(\xi)\,d\xi.$

[Sneddon I. M., 1972]

Infact, to all intents and purposes this is considered as the most successful and authentically approved compilation of open problems ever to be produced by an individual mathematician.

Hilbert had this to say:

> Who among us would not be happy to lift the veil behind which is hidden the future; to gaze at the coming developments of our science and at the secrets of its development in the centuries to come? What will be the ends toward which the spirit of future generations of mathematicians will tend? What methods, what new facts will the new century reveal in the vast and rich field of mathematical thought?

The Hilbert-Schmidt solution of a Hermitian Integral Equation is of the form $y = f + \lambda \sum_{1}^{\infty} \frac{f'}{\lambda_p - \lambda} u_p(x).$

[Murnaghan F. D., 1948]

Hilbert attempted to build up mathematics by using symbolic logic in a way that would prove the consistency of Mathematics. His approach was dealt a mortal blow by Kurt Gödel (1906-1978) who proved that there will always be "undecidable" problems in any sufficiently rich axiomatic system; that is, that in any mathematical system of any consequence, there will always be statements that can never be proven either true or false. [Judson T. W., 1994]

The non-countably infinite dimensional space which consist of all functions (vectors), $f(x)$, $a \le x \le b$, which have the finite norm (length) is often called Hilbert Space after David Hilbert who investigated the properties of such functions. [Spiegel M. R., 1981]

Hilbert unified the field of algebraic number theory with his 1897 treatise Zahlbericht (literally "report on numbers"). He also resolved a significant number-theory problem formulated by Waring in 1770. As with the finiteness theorem, he used an existence proof that shows there must be solutions for the problem rather than providing a mechanism to produce the answers. He then had little more to publish on the subject; but the emergence of Hilbert modular forms in the dissertation of a student means his name is further attached to a major area. [Ewald, W. B.,1996.]

Added to that, in 1920 he proposed explicitly a research project that became known as Hilbert's program. He wanted mathematics to be formulated on a solid and complete logical foundation. He believed that in principle this could be done, by showing that:

1. all of mathematics follows from a correctly chosen finite system of axioms; and
2. that some such axiom system is provably consistent through some means such as the epsilon calculus.

This approach has been successful and influential in relation with Hilbert's work in algebra and functional analysis, but has failed to engage in the same way with his interests in physics and logic. He made a series of conjectures on class field theory. The concepts were highly influential, and his own contribution lives on in the names of the Hilbert class field and of the Hilbert symbol of local class field theory. Results on them were mostly proved by 1930, after work by Teiji Takagi. Hilbert did not work in the central areas of analytic number theory, but his name has become known for the Hilbert–Pólya conjecture, for reasons that are anecdotal. [Ewald, W. B.,1996.]

4.0 The Works of Hilbert

Hilbert wrote in 1919:

> We are not speaking here of arbitrariness in any sense. Mathematics is not like a game whose tasks are determined by arbitrarily stipulated rules. Rather, it is a conceptual system possessing internal necessity that can only be so and by no means otherwise.

Hilbert published his views on the foundations of mathematics in the 2-volume work Grundlagen der Mathematik. David Hilbert's work threw light on its application (Obeng-Denteh, 2011).

Another aspect which is of imperative concern is that around 1909, Hilbert dedicated himself to the study of differential and integral equations; his work had direct consequences for important parts of modern functional analysis. In order to carry out these studies, Hilbert introduced the concept of an infinite dimensional Euclidean space, later called Hilbert space. His work in this part of analysis provided the basis for important contributions to the mathematics of physics in the next two decades, though from an unanticipated direction. Later on, Stefan Banach amplified the concept, defining Banach spaces. Hilbert space is the most important single idea in the area of functional analysis, particularly of the spectral theory of self-adjoint linear operators, that grew up around it during the 20th century.

Until 1912, Hilbert was almost exclusively a "pure" mathematician. When planning a visit from Bonn, where he was immersed in studying physics, his fellow mathematician and friend Hermann Minkowski joked he had to spend 10 days in quarantine before being able to visit Hilbert. In fact, Minkowski seems responsible for most of Hilbert's physics investigations prior to 1912, including their joint seminar in the subject in 1905.

In 1912, three years after his friend's death, Hilbert turned his focus to the subject almost exclusively. He made moves to have a "physics tutor" for himself. He started studying kinetic gas theory and moved on to elementary radiation theory and the molecular theory of matter. Even after the war started in 1914, he continued seminars and classes where the works of Albert Einstein and others were followed closely.

By 1907 Einstein had framed the fundamentals of the theory of gravity, but then struggled for nearly 8 years with a confounding problem of putting the theory into final form.[26] By early summer 1915, Hilbert's interest in physics had focused him on general relativity, and in organizing a week of lectures on the subject he invited Einstein to Göttingen to speak on it. An enthusiastic reception was accorded Einstein at Göttingen. Over the summer Einstein learned that Hilbert was also working on the field equations and redoubled his own efforts. During November 1915 Einstein published several papers culminating in "The Field Equations of Gravitation". Nearly simultaneously David Hilbert published "The Foundations of Physics", an axiomatic derivation of the field equations. Hilbert fully credited Einstein as the originator of the theory, and no public priority dispute concerning the field equations ever arose between the two men during their lives.

Additionally, Hilbert's work anticipated and assisted several advances in the mathematical formulation of quantum mechanics. His work was a key aspect of Hermann Weyl and John von Neumann's work on the mathematical equivalence of Werner Heisenberg's matrix mechanics and Erwin Schrödinger's wave equation and his namesake Hilbert space plays an important part in quantum theory. In 1926 von Neuman showed that if atomic states were understood as vectors in Hilbert space, then they would correspond with both Schrödinger's wave function theory and Heisenberg's matrices.

5.0 Hilbert's Unification of Algebraic Number Theory

Hilbert unified the field of algebraic number theory with his 1897 treatise Zahlbericht which literally means "report on numbers". He made a series of conjectures on class field theory. The concepts were highly influential, and his own contribution lives on in the names of the Hilbert class field and of the Hilbert symbol of local class field theory. Hilbert did not work in the central areas of analytic number theory, but his name has become known for the Hilbert–Pólya conjecture, for reasons that are anecdotal (David Hilbert, Wiki).

Hilbert's Nullstellensatz is a German expression which succinctly means "theorem of zeros," or more literally, "zero-locus-theorem". This is a theorem which makes precise a fundamental relationship between the geometric and algebraic sides of algebraic geometry. It is an important branch of mathematics which relates algebraic sets to ideals in polynomial rings over algebraically closed fields. It was first proved by David Hilbert whom it is named after.

The classical existence theorem of Hilbert has many precise analogues, strengthenings, and generalizations. Since 'Stellen' means 'places' in German, one sees immediately that the content is the geometrical one of the existence of places in a space which satisfy given conditions. Because the conditions considered are equations between functions defined on the space, the geometry is intimately related to the algebra of such functions. However, to speak of zeroes (Nullstellen) is an unnecessary restriction, useful however in limited contexts where there are theorems available concerning factorization et cetera. The question of existence and partial answers also make sense for 'rings' without necessarily an everywhere-defined subtraction, for example in the natural numbers or in real algebraic geometry where one seeks 'Positivensaetze'. The usefulness of the restriction to algebras with negatives led to the development of the technique of ideal theory, in particular to the study of the generation of the unit ideal, et cetera. However, from a more conceptual perspective the purpose of the ideals is to give rise to quotient algebras. In that light one sees that the more natural algebraic interpretation of closed subset is as a surjective algebra homomorphism from the algebra of functions on a space to another algebra; in the same spirit the role of points i.e. the desired Stellen, is to act as general 'evaluation' homomorphisms from the same algebra to special algebras. The concern about maximal ideals and prime ideals comes really from the question of which algebras are special. The general idea is that the special algebras can be qualitatively smaller than the typical algebras, but such homomorphisms can be proved to exist nonetheless (What makes a theorem a nullstellensatz,mathoverflow.net)

6.0 Explanation of the Hilbert's Nullstellensatz

Emphasis is placed on Hilbert's contribution to the Nullstellensatz, the explanation is as follows:

Let k be a field such as the rational numbers and K be an algebraically closed field extension such as the complex numbers, consider the polynomial ring $k[X_1,X_2,..., X_n]$ and let I be an ideal in this ring. In ring theory, a branch of abstract algebra, an ideal is a special subset of a ring. The ideal concept allows the generalization in an appropriate way of some important properties of integers like "even number" or "multiple of 3".

For instance, in rings one studies prime ideals instead of prime numbers, one defines coprime ideals as a generalization of coprime numbers, and one can prove a generalized Chinese remainder theorem about ideals. In a certain class of rings important in number theory, the Dedekind domains, one can even recover a version of the fundamental theorem of arithmetic: in these rings, every nonzero ideal can be uniquely written as a product of prime ideals. If p is in R, then pR is a right ideal and Rp is a left ideal of R, the set of real numbers. Examples of ideals are:

- The even integers form an ideal in the ring Z of all integers; it is usually denoted by 2Z. This is because the sum of any even integers is even, and the product of any integer with an even integer is also even. Similarly, the set of all integers divisible by a fixed integer n is an ideal denoted nZ.
- The set of all polynomials with real coefficients which are divisible by the polynomial $x^2 + 1$ is an ideal in the ring of all polynomials.
- The set of all n-by-n matrices whose last row is zero forms a right ideal in the ring of all n-by-n matrices. It is not a left ideal. The set of all n-by-n matrices whose last column is zero forms a left ideal but not a right ideal.

- The ring C(R) of all continuous functions f from R to R contains the ideal of all continuous functions f such that f(1) = 0. Another ideal in C(R) is given by those functions which vanish for large enough arguments, i.e. those continuous functions f for which there exists a number L > 0 such that f(x) = 0 whenever |x| > L.

The affine variety V(I) defined by this ideal consists of all n-tuples $x = (x_1,...,x_n)$ in K^n such that f(x) = 0 for all f in I. Hilbert's Nullstellensatz states that if p is some polynomial in $k[X_1,X_2,..., X_n]$ which vanishes on the variety V(I), i.e. p(x) = 0 for all x in V(I), then there exists a natural number r such that p^r is in I.

An immediate corollary is the "weak Nullstellensatz": The ideal I in $k[X_1,X_2,..., X_n]$ contains 1 if and only if the polynomials in I do not have any common zeros in K^n.

When k = K the "weak Nullstellensatz" may also be stated as follows: if I is a proper ideal in $K[X_1,X_2,..., X_n]$, then V(I) cannot be empty, i.e. there exists a common zero for all the polynomials in the ideal. This is the reason for the name of the theorem, which can be proved easily from the 'weak' form using the Rabinowitsch trick. The assumption that K be algebraically closed is essential here; the elements of the proper ideal $(X^2 + 1)$ in R[X] do not have a common zero. The weak Nullstellensatz is a generalization of the fundamental theorem of algebra, which is the case n = 1, I in this case being the proper ideal generated by the polynomial, and V(I) being the zeros.

With the notation common in algebraic geometry, the Nullstellensatz can also be formulated as

$$\mathrm{I}(\mathrm{V}(J)) = \sqrt{J}$$

for every ideal J. Here, $\sqrt{J}$ denotes the radical of J and I(U) is the ideal of all polynomials which vanish on the set U. Examples of radicals are:

Consider the ring Z of integers.

1. The radical of the ideal 6Z of integer multiples of 6 is 6Z.
2. The radical of 7Z is 7Z.
3. The radical of 12Z is 6Z.
4. In general, the radical of nZ is rZ, where r is the product of all prime factors of n.

In this way, we obtain an order-reversing bijective correspondence between the affine varieties in K^n and the radical ideals of $K[X_1,X_2,..., X_n]$. In fact, more generally, one has a Galois connection between subsets of the space and subsets of the algebra, where "Zariski closure" and "radical of the ideal generated" are the closure operators.

The fact that commuting matrices have a common eigenvector – and hence by induction stabilize a common flag and are simultaneously triangularizable – can be interpreted as a result of the weak Nullstellensatz (Nullstellensatz, Wiki).

7.0 Conclusion

The study in no small pedestal has reconfirmed the historical contributions of David Hilbert in the area mathematics. It emerged that he contributed to the growth of especially the Nullstellensatz and was the first person to prove it. Information gleaned from various sources bear eloquent testimony to the said notion. The implications and conclusions are that the current trend in the said area of mathematics could be appreciated owing to contributions and motivations of such a mathematician.

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