A Representation of Analytic Functions in Lower Half Complex Plane

Lazim Kamberi

Depatment of Mathrmatica, State University of Tetovo, Macedonia

Shpëtim Rexhepi

Depatment of Mathrmatica, State University of Tetovo, Macedonia

Bilall Shaini

Depatment of Mathrmatica, State University of Tetovo, Macedonia

Abstract

In this paper in the introduction part we will give the definition of Hardy Spaces, analytic functions defined in Hardy Spaces, the existence of boundary functions defined as the limit of analytic functions defined in Hardy Spaces when the imaginary part approaches to left side of zero, the class of rapidly decreasing infinitely differentiable functions and in the second part of the paper in the main result we will state a theorem which describes the representation of some analytic functions with distributions in the lower complex half plane and give its proof .

Keywords : analytic function, Hardy Spaces, boundary function, tempered distribution

1. Introduction

We denote by $\prod^-$ the lower half complex plane i.e. $\prod^- = \{z \mid \operatorname{Im} z < 0\}$.

An analytic function is a function that is locally given by convergent power series. Suppose that f is an analytic function on the lower half complex plane and $p > 0, p \in R$. We say that $f \in H^p\left(\prod^-\right)$ if $\sup\limits_{y<0} \int\limits_R \left| f(x+iy) \right|^p \le C$ and f belongs to $H^\infty\left(\prod^-\right)$ if $\sup\limits_{y<0} \left| f(x+iy) \right| \le c$ almost everywhere on R. It is known that $H^\infty \subset H^p$.

The H^p spaces are called ***Hardy spaces*****.**

If $f \in H^p\left(\prod^-\right)$ then the boundary function $f^*(x) = \lim\limits_{y\to 0^-} f(x+iy)$ exists almost everywhere,

$f^* \in L^p = L^p(R)$ and $\int\limits_R \frac{\log\left|f^*(x)\right|}{1+x^2} dx > -\infty.$

Let $S(R)$ be the class of rapidly decreasing infinitely differentiable functions and $f \in H^p(C \setminus R)$. Than $g \in S(R)$ the limit $A(g) = \lim\limits_{y\to 0^-} \int\limits_R \left[f(x-iy) - f(x+iy) \right] g(x)\,dx$ exist and is tempered distribution.

2. Main result

In this part we will prove the following theorem which describes the representation of some analytic functions with distributions.

Theorem: Let f be an analytic and bounded function on $C \setminus R$, continous on the real line R and which has no zeros on R.

Then, for $g \in S(R)$, it holds $A_f(g) = 2\int_R g(x)\log\left(\frac{1}{\left|f^*(t)\right|}\right)dx$,

where $A_f(g) = \lim_{y\to 0^-}\int_R [f(x-iy) - f(x+iy)]g(x)dx$, $z = x+iy \in \Pi^-$ and $f^*(x)$ is the boundary values of the function f.

Proof.

Using the representation in $[1]$, we have that f is of the form $f(z) = \exp\left(\frac{1}{\pi i}\int_R \frac{(1+tz)\log\left|f^*(t)\right|}{(t-z)(1+t^2)}dt\right)$, where $f^*(t)$ is boundary value of the function f. For $g \in S(R)$, we have

$$A_f(g) = \lim_{y\to 0^-}\int_R [f(x-iy) - f(x+iy)]g(x)dx =$$

$$= \lim_{y\to 0^-}\int_R \left[\exp\left(\frac{1}{\pi i}\int_R \frac{(1+t\bar{z})\log\left|f^*(t)\right|}{(1-\bar{z})(1+t^2)}dt\right) - \exp\left(\frac{1}{\pi i}\int_R \frac{(1+tz)\log\left|f^*(t)\right|}{(t-z)(1+t^2)}dt\right)\right]g(x)dx.$$

We take the partition of the segment $[-M, M]$ in to equal parts $\tau = \{-M = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = M\}$, $\Delta t_k = \frac{2M}{n}$, we have $\exp\left(\frac{1}{\pi i}\int_R \frac{(1+t\bar{z})\log\left|f^*(t)\right|}{(t-\bar{z})(1+t^2)}dt\right) = \lim_{M\to\infty}\lim_{n\to\infty}\exp\left(\frac{1}{\pi i}\sum_{k=0}^{n}\frac{(1+t_k\bar{z})\log\left|f^*(t_k)\right|}{(t_k-\bar{z})(1+t_k^2)}\frac{2M}{n}\right)$.

Using this relation and the Taylor series development, we have:

$$B=\exp\left(\frac{1}{\pi i}\int_R\frac{(1+t\bar{z})\log\left|f^*(t)\right|}{(t-\bar{z})(1+t^2)}dt\right)-\exp\left(\frac{1}{\pi i}\int_R\frac{(1+tz)\log\left|f^*(t)\right|}{(t-z)(1+t^2)}dt\right)=$$

$$=\lim_{M\to\infty}\lim_{n\to\infty}\left[\exp\left(\frac{1}{\pi i}\sum_{k=0}^{n}\frac{(1+t_k\bar{z})\log\left|f^*(t_k)\right|}{(t_k-\bar{z})(1+t_k^2)}\frac{2M}{n}\right)-\left(\frac{1}{\pi i}\sum_{k=0}^{n}\frac{(1+t_kz)\log\left|f^*(t_k)\right|}{(t_k-z)(1+t_k^2)}\frac{2M}{n}\right)\right]=$$

$$=\lim_{M\to\infty}\lim_{n\to\infty}\left[1+\left(\frac{1}{\pi i}\sum_{k=0}^{n}\frac{(1+t_k\bar{z})\log\left|f^*(t_k)\right|}{(t_k-\bar{z})(1+t_k^2)}\frac{2M}{n}\right)+\frac{1}{\underset{n\to\infty}{2!}}\left(\frac{1}{\pi i}\sum_{k=0}^{n}\frac{(1+t_k\bar{z})\log\left|f^*(t_k)\right|}{(t_k-z)(1+t_k^2)}\frac{2M}{n}\right)^2+\ldots-\right.$$

$$\left.-1-\left(\frac{1}{\pi i}\sum_{k=0}^{n}\frac{(1+t_kz)\log\left|f^*(t_k)\right|}{(t_k-z)(1+t_k^2)}\frac{2M}{n}\right)-\frac{1}{2!}\left(\frac{1}{\pi i}\sum_{k=0}^{n}\frac{(1+t_kz)\left|f^*(t_k)\right|}{(t_k-z)(1+t_k^2)}\frac{2M}{n}\right)^2-\ldots\right]=$$

$$=\lim_{M\to\infty}\lim_{n\to\infty}\left[\left(\frac{1}{\pi i}\sum_{k=0}^{n}\frac{(1+t_k\bar{z})\log\left|f^*(t_k)\right|}{(t_k-\bar{z})(1+t_k^2)}\frac{2M}{n}\right)-\left(\frac{1}{\pi i}\sum_{k=0}^{n}\frac{(1+t_kz)\log\left|f^*(t_k)\right|}{(t_k-z)(1+t_k^2)}\frac{2M}{n}\right)+R(\bar{z},z,t)\right].$$

Where $R(\bar{z},z,t)=\frac{1}{2!}\left(\frac{1}{\pi i}\sum_{k=0}^{n}\frac{(1+t_k\bar{z})\log\left|f^*(t_k)\right|}{(t_k-\bar{z})(1+t_k^2)}\frac{2M}{n}\right)^2+\ldots-\frac{1}{2!}\left(\frac{1}{\pi i}\sum_{k=0}^{n}\frac{(1+t_k\bar{z})\log\left|f^*(t_k)\right|}{(t_k-z)(1+t_k^2)}\frac{2M}{n}\right)^2-\ldots.$

Since $\frac{1+t_k\bar{z}}{(t_k-\bar{z})(1+t_k^2)}-\frac{1+t_kz}{(t_k-z)(1+t_k^2)}=\frac{t_k+t_k^2\bar{z}-z-t_kz\bar{z}-t_k-t_k^2z+\bar{z}+t_kz\bar{z}}{(t_k-\bar{z})(t_k-z)(1+t_k^2)}=$

$=\frac{\bar{z}-z}{(t_k-x+iy)(t_k+x-iy)}=\frac{-2yi}{(x-t_k)^2+y^2}$, we have

$$B=\exp\left(\frac{1}{\pi i}\int_R\frac{(1+t\bar{z})\log\left|f^*(t)\right|}{(t-\bar{z})(1+t^2)}dt\right)-\exp\left(\frac{1}{\pi i}\int_R\frac{(1+tz)\log\left|f^*(t)\right|}{(t-z)(1+t^2)}dt\right)=$$

$$=\lim_{M\to\infty}\lim_{n\to\infty}\left(-\frac{2}{\pi}\sum_{k=0}^{n}\frac{y\log\left|f^*(t_k)\right|}{(x-t_k)^2+y^2}\frac{2M}{n}+R(\bar{z},z,t)\right).$$

Using that $\lim_{M\to\infty}\lim_{n\to\infty}R(\bar{z},z,t)=0$, we get that

$B=\lim_{M\to\infty}\lim_{n\to\infty}\left(-\frac{2}{\pi}\sum_{k=0}^{n}\frac{y\log\left|f^*(t_k)\right|}{(x-t_k)^2+y^2}\frac{2M}{n}\right)$. So

$$A_f(g) = \lim_{y\to 0^-}\int_R [f(x-iy) - f(x+iy)]g(x)dx = \lim_{y\to 0^-}\int_R Bg(x)dx =$$

$$= \lim_{y\to 0^-}\int_R \left[\lim_{M\to\infty}\lim_{n\to\infty}\left(-\frac{2}{\pi}\sum_{k=0}^{n}\frac{y\log\left|f^*(t_k)\right|}{(x-t_k)^2+y^2}\frac{2M}{n}\right)\right]g(x)\,dx =$$

$$= \lim_{y\to 0^-}\int_R \left[-\frac{2}{\pi}\int_R P(x-t,y)\log\left|f^*(t)\,dt\right|\right]gxdx =$$

$\lim_{y\to 0^-}\left[\frac{2}{\pi}\int_R P(x-t,y)\log\left(\frac{1}{\left|f^*(t)\right|}\right)\right]g(x)dx = 2\int_R g(x)\log\left(\frac{1}{\left|f^*(x)\right|}\right)dx$, wheve $P(x-t,y)$ is the Poisson kernel in the summation of the series.

References

[1] Duren, P.L, Theory of H^p spaces, Ac. Press, New York, 1970

[2] A.I. Stepants, Methods of Approximation Theory, Netherland 2005

[3] G.H Hardy, W.W. Rogosinski, Fouries Series, Cambridge 1996

[4] Bremerman, G, Distributions, Complex variables and Fourier transforms, Addison – Westay, Massachusets 1965.

[5] Cima J, Ross W. The backward shift on the Hardy space, AMS Providenca, 2000.