Commutation Relations for the Powers of the Bosonic Operator

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Abstract

In this paper, a general formula for the commutation relation is obtained for the powers of the bosonic creation and annihilation operators using the combinatorial methods. By using the usual commutation relation $[a,a^{\dagger}]=aa^{\dagger}-a^{\dagger}a=1$, our formula is obtained in terms of the number operator. Finally, the explicit expressions of the commutators for (s = 2, 3 and 4) considered recently by Chair in [Chair N., (2013). The Euler-Riemann Gases, and partition Identities. Nuclear Physics B, 872: 72-105.] are checked.

Keywords: non-interacting quantum field theory; commutation relations; bosonic operators; combinatorial methods.

Introduction

The simple harmonic oscillator is one of the most important problems in quantum mechanics. It doesn't only illustrate many of the basic concepts and methods of quantum mechanics but it also can approximate essentially any potential well (Sakurai and Napolitano, 2011), so it describes phenomena from molecular vibrations to nuclear structure. Moreover, because the Hamiltonian is a sum of squares of two canonically conjugate variables, it is also an important starting point for much of quantum field theory.

In order to connect the non-interacting quantum field theories with multiplicative number theory the spectrum of the harmonic oscillator is chosen to be logarithmic (Julia, 1989; Spector, 1998), in these theories the partition functions are given in terms of the Riemann zeta functions or other Dirichlet series. Chair in (Chair, 2013) considered non- interacting quantum field theories without logarithmic spectrum to make a connection with the additive number theory, in these theories the partition functions are given in terms of the Euler generating function in partition theory (Chair, 2013). Spector in (Spector, 1998) assumed that the square and higher powers of the bosonic operators are bosonic operators, this assumption turns out to be false, and was corrected later by Chair (Chair, 2013). Recently Alsharafat in (Alsharafat and Chair, 2017) considered non- interacting quantum field theories and used the Path integral formulation to obtain a general factorization formula that relates the bosonic and fermionic partition functions of the harmonic Oscillator.

In this paper, we used a combinatorics methods (Charalambos, 2002) to obtain a general expression for the commutator $[(a)^{l},(a^{\dagger})^{m}]$ for the bosonic operators, where $l \leq m$, and a general formula of the commutator $[(a)^{s},(a^{\dagger})^{s}]$ is obtained in terms of the number operator $a^{\dagger}a=\mathbf{n}$. Then, the explicit expressions of the commutation relation for the powers ($s=2$, $s=3$ and $s=4$) of the bosonic operators that obtained in (Chair, 2013) are checked. As a consequence of the general expression for the commutator of the bosonic operators, the well-known consequence says that the square and higher powers of the bosonic operators are not bosonic operators has been verified.

Our paper is organized as follows. In section 2, we used a combinatorics methods to obtain a general expression for the commutator $[(a)^l,(a^\dagger)^m]$ for the powers of the bosonic operators. In section 3, we obtain a general expression for the commutator $[(a)^s,(a^\dagger)^s]$ for the powers of the bosonic operators in terms of the number operator $a^\dagger a=\mathbf{n}$ and we checked that for (s = 2, 3 and 4) they reduced to the commutators that are given in (Chair, 2013). In section 4, our conclusion is given.

Commutation Relations for the Powers of the Bosonic Operators.

In this section, we used a combinatorial methods to obtain a general expression for the commutation relations for the powers of the bosonic operators, i.e., the commutators $[(a)^l,(a^\dagger)^m]$. Our starting point is the well-known commutation relations of the creation and annihilation operators for noninteracting bosonic harmonic oscillators (Feynman, 1982).

$$\left[a_i,a_j^\dagger\right]=\delta_{ij} \quad , \tag{1}$$

where $a_j^\dagger\left(a_i\right)$ are the bosonic creations (annihilation) operators, respectively. The ground state $\left|0\right\rangle$ is defined by $a_i\left|0\right\rangle=0$ for all i. The quantum states $\left|n\right\rangle$ are obtained by acting with the creation operators in the ground state, i.e., $\left|n\right\rangle=\left(a_1^\dagger\right)^{n_1}\left(a_2^\dagger\right)^{n_2}...\left(a_k^\dagger\right)^{n_k}...\left|0\right\rangle$. The number operator $n=\sum_{k=1}^{\infty}ka_k^\dagger a_k$ acting on the state $\left|n\right\rangle$ such that $n\left|n\right\rangle=n\left|n\right\rangle$, where $n=\sum_{k=1}^{\infty}kn_k$.

For single oscillator – in this work, we restrict our self in this case- Chair in (Chair, 2013) gives the following explicit expressions of the commutators $[(a)^s,(a^\dagger)^s]$ for (s = 2, s = 3 and s = 4) as:

$$\left[(a)^2,\left(a^\dagger\right)^2\right]=4a^\dagger a+2 \tag{2}$$

$$\left[(a)^3,\left(a^\dagger\right)^3\right]=9a^\dagger aa^\dagger a+9a^\dagger a+6 \tag{3}$$

$$\left[(a)^4,\left(a^\dagger\right)^4\right]=16a^\dagger aa^\dagger aa^\dagger a+24a^\dagger aa^\dagger a+56a^\dagger a+24 \quad , \tag{4}$$

where $a^\dagger\left(a\right)$ are the bosonic creations (annihilation) operators, respectively. Which are satisfy the usual commutators given by (Feynman, 1982):

$$\begin{aligned}&[a,a^\dagger]=1,\\&[a^\dagger,a^\dagger]=0\\&[a,a]=0\end{aligned} \tag{5}$$

The ground state $\left|0\right\rangle$ is defined by $a\left|0\right\rangle=0$, the quantum state denoted by $\left|n\right\rangle$ is obtained by acting the creation operators on the ground state, i.e., $\left|n\right\rangle=\left(a^\dagger\right)^n\left|0\right\rangle$. Using the combinatorial methods and Eqs. (5), one can easy obtain:

$$[a,(a^\dagger)^2]=2a^\dagger \tag{6}$$

$$[a,(a^\dagger)^3]=3(a^\dagger)^2 \tag{7}$$

$$\left[a,(a^\dagger)^4\right]=4(a^\dagger)^3 \tag{8}$$

The a bove equations can be written in a general form as

$$\left[a,\left(a^\dagger\right)^m\right]=m\left(a^\dagger\right)^{m-1}. \tag{9}$$

Using the same method, we find that

$$\left[(a)^2,\left(a^\dagger\right)^m\right]=2m\left(a^\dagger\right)^{m-1}a+m\left(m-1\right)\left(a^\dagger\right)^{m-2} \tag{10}$$

and

$$\left[(a)^3,\left(a^\dagger\right)^m\right]=3m\left(a^\dagger\right)^{m-1}(a)^2+3m(m-1)\left(a^\dagger\right)^{m-2}a+m(m-1)(m-2)\left(a^\dagger\right)^{m-3} \tag{11}$$

For general commutation relation for the power of bosonic creation and annihilation operators, we propose the general formula, for ($l\le m$) as the following

$$\left[(a)^l,\left(a^\dagger\right)^m\right]=\sum_{k=1}^{l}P_k^l C_k^m\left(a^\dagger\right)^{m-k}(a)^{l-k} \tag{12}$$

where $a^\dagger(a)$, are the bosonic creation (annihilation) operators, respectively. P_k^l corresponds to arrangements of k objects among l objects and C_k^m are the binomial coefficients (Charalambos, 2002), which are given by respectively,

$$P_k^l=\frac{l!}{\left(l-k\right)!} \quad , \tag{13}$$

$$C_k^m=\frac{m!}{k!\left(m-k\right)!} \quad , \tag{14}$$

It is clear that our proposed formula, namely, Eq. (12) represent a generating function which is encoding an infinite sequences of numbers (P_k^l and C_k^m) by treating them as the coefficients of a series of power operators. To check our propsed formula for the commutators given by Eq. (12), we give the following example, for $l=3$, in this case Eq. (12) can be written as:

$$\left[(a)^3,\left(a^\dagger\right)^m\right]=\sum_{k=1}^{l=3}P_k^3 C_k^m\left(a^\dagger\right)^{m-k}(a)^{3-k}$$

$$=P_1^3C_1^m\left(a^\dagger\right)^{m-1}a^2+P_2^3C_2^m\left(a^\dagger\right)^{m-2}a+P_3^3C_3^m\left(a^\dagger\right)^{m-3}$$

$$\left[(a)^3,\left(a^\dagger\right)^m\right]=3m\left(a^\dagger\right)^{m-1}(a)^2+3m\left(m-1\right)\left(a^\dagger\right)^{m-2}a+m\left(m-1\right)\left(m-2\right)\left(a^\dagger\right)^{m-3} \tag{15}$$

It is clear that for $l=3$, Eq. (12) reduced to Eq. (11).

Now, we take a special case when ($l=m$), our formula is read as:

$$\left[(a)^m,\left(a^\dagger\right)^m\right]=\sum_{k=1}^{m}(C_k^m)^2k!\left(a^\dagger\right)^{m-k}(a)^{m-k}. \tag{16}$$

where $a^\dagger(a)$, are the bosonic creation (annihilation) operators, respectively, and C_k^m are the binomial coefficients are given by Eq.(14). Consider the following simple example for ($l=m=4$), Eq. (16) can be written as:

$$\left[(a)^4,(a^\dagger)^4\right]=\sum_{k=1}^{4}(C_k^4)^2k!\left(a^\dagger\right)^{4-k}(a)^{4-k}$$

$$=(C_1^4)^2\left(a^\dagger\right)^3a^3+(C_2^4)^2 2!\left(a^\dagger\right)^2a^2+(C_3^4)^2 3!\left(a^\dagger\right)a+(C_4^4)^2 4!$$

$$\left[(a)^4,(a^\dagger)^4\right]=16\left(a^\dagger\right)^3(a)^3+72\left(a^\dagger\right)^2(a)^2+96a^\dagger a+24 \qquad (17)$$

It is clear that the above equation - which is a special case from Eq. (16) when ($l=m=4$) - is the same as the commutator obtained in (Chair, 2013), i.e., Eq. (4).

In order to develop our general formula for commutation relations for the powers of the bosonic operators, we recall that the short-hand notation for the non-commuting operators $\left[A,B\right]=AB-BA$, the Eq. (12) may be written as

$$\begin{aligned}\left[(a)^l,(a^\dagger)^m\right]&=a^l(a^\dagger)^m-(a^\dagger)^m a^l\\ &=\sum_{k=1}^{l}P_k^lC_k^m\left(a^\dagger\right)^{m-k}(a)^{l-k}\end{aligned} \qquad (18)$$

the above equation may be written as :

$$a^l(a^\dagger)^m=(a^\dagger)^m a^l+\sum_{k=1}^{l}P_k^lC_k^m\left(a^\dagger\right)^{m-k}(a)^{l-k}$$

$$(a)^l\left(a^\dagger\right)^m=\sum_{k=0}^{l}P_k^lC_k^m\left(a^\dagger\right)^{m-k}(a)^{l-k} \quad , \qquad (19)$$

where $a^\dagger(a)$, are the bosonic creation (annihilation) operators, respectively, P_k^l corresponds to arrangements of k objects among l objects and C_k^m are the binomial coefficients, which are given by Eq.(13) and Eq.(14) respectively, $(l\le m)$. For special case when $(l=m)$, Eq. (19) can be written as

$$(a)^m\left(a^\dagger\right)^m=\sum_{k=0}^{m}(C_k^m)^2k!\left(a^\dagger\right)^{m-k}(a)^{m-k} \quad , \qquad (20)$$

the above equaton is another form of Eq.(12), also it is represent a generating function which is encoding an infinte sequence of number $(C_k^m)^2k!$ by treating them as the coefficients of a series of power operators.

Commutation Relations for Powers of Bosonic Operators and the Operator Number.

In this section, we used a combinatorial methods to obtain a general expression of the commutation relations for the powers of the bosonic operators in terms of the number operator. Our derivations are states from the usual commutators $\left[a,a^\dagger\right]=aa^\dagger-a^\dagger a=1$ and the number operator $\mathbf{n}=a^\dagger a$. One can easly obtain the following expressions,

$$\left(a^{\dagger}\right)^{2}\left(a\right)^{2}=a^{\dagger}aa^{\dagger}a-a^{\dagger}a=\mathbf{n}(\mathbf{n}-1) \qquad (21)$$

$$\left(a^{\dagger}\right)^{3}\left(a\right)^{3}=\mathbf{n}\left(\mathbf{n}-1\right)\left(\mathbf{n}-2\right) \qquad (22)$$

$$\left(a^{\dagger}\right)^{4}\left(a\right)^{4}=\mathbf{n}(\mathbf{n}-1)(\mathbf{n}-2)(\mathbf{n}-3) \qquad (23)$$

Next, we propose a formula for the powers of the bosonic operators for any order s . The formula we propose is given by the following expression,

$$\left(a^{\dagger}\right)^{s}\left(a\right)^{s}=\mathbf{n}(\mathbf{n}-1)(\mathbf{n}-2)(\mathbf{n}-3)...(\mathbf{n}-s-1) \qquad (24)$$

the above equation may be written in general form as:

$$\left(a^{\dagger}\right)^{s}\left(a\right)^{s}=\prod_{j=0}^{s-1}(\mathbf{n}-j) \qquad (25)$$

where $a^{\dagger}(a)$, are the bosonic creation (annihilation) operators, respectively, and $\mathbf{n}$ is the operate number.

Here, we would like to connect the general expression for the cmmutation relations for the powers of the bosonic operators, namely Eq. (16) with the number operator $\mathbf{n}$, inserting Eq. (25) into Eq. (16), one can find that:

$$\left[\left(a\right)^{s},\left(a^{\dagger}\right)^{s}\right]=\sum_{k=1}^{s}(C_{k}^{s})^{2}k!\prod_{j=0}^{(s-1)-k}(\mathbf{n}-j) \quad , \qquad (26)$$

where $a^{\dagger}(a)$, are the bosonic creation (annihilation) operators, respectively, C_{k}^{s} are the binomial coefficients defined in Eq.(14) and $\mathbf{n}=a^{\dagger}a$ is the operate number. Note that, we set $\prod_{j=0}^{(s-1)-k}(\mathbf{n}-j)=1$ for $k>(s-1)$ and $\prod_{j=0}^{(s-1)-k}(\mathbf{n}-j)=\mathbf{n}$ for $k=(s-1)$.

One can easily check Eq. (26) by taking some examples, for $s=3$, we have

$$\left[\left(a\right)^{3},\left(a^{\dagger}\right)^{3}\right]=\sum_{k=1}^{3}(C_{k}^{3})^{2}k!\prod_{j=0}^{2-k}(\mathbf{n}-j)$$

$$=(C_{1}^{3})^{2}\prod_{j=0}^{1}(\mathbf{n}-j)+2(C_{2}^{3})^{2}\mathbf{n}+6(C_{3}^{3})^{2}$$

$$=9\mathbf{n}^{2}+9\mathbf{n}+6$$

$$\left[\left(a\right)^{3},\left(a^{\dagger}\right)^{3}\right]=9a^{\dagger}aa^{\dagger}a+9a^{\dagger}a+6 \qquad (27)$$

This is exactly the explicit expression of the commutator $\left[\left(a\right)^{3},\left(a^{\dagger}\right)^{3}\right]$ obtained by Chair in (Chair, 2013), namely Eq. (3), and for $s=4$, we have

$$\left[\left(a\right)^{4},\left(a^{\dagger}\right)^{4}\right]=\sum_{k=1}^{4}(C_{k}^{4})^{2}k!\prod_{j=0}^{3-k}(\mathbf{n}-j)$$

$$=(C_{1}^{4})^{2}\prod_{j=0}^{2}(\mathbf{n}-j)+2(C_{2}^{4})^{2}\prod_{j=0}^{1}(\mathbf{n}-j)+6(C_{3}^{4})^{2}\mathbf{n}+12(C_{4}^{4})^{2}$$

$$=16\mathbf{n}^{3}+24\mathbf{n}^{2}+56\mathbf{n}+24$$

$$\left[(a)^4,(a^\dagger)^4\right]=16a^\dagger aa^\dagger aa^\dagger a+24a^\dagger aa^\dagger a+56a^\dagger a+24 \tag{28}$$

where $\mathbf{n}^3=a^\dagger aa^\dagger aa^\dagger a$ and $\mathbf{n}^2=a^\dagger aa^\dagger a$, also it is clear that the above equation is exactly the explicit expression of the commutator $\left[(a)^4,(a^\dagger)^4\right]$ obtained by Chair in (Chair, 2013), namely, Eq.(4).

From Eq. (26) and its consequences, it is clear that our results are in agreement with the the well known consequence says that the square and higher powers of the bosonic operators are not bosonic operators.

Conclusion

In this paper, we used the combinatorial methods to obtain a general formula for the powers of the bosonic creation and annihilation operators. This formula works as a generating function, which is encoding an infinite sequences of numbers (and) by treating them as the coefficients of a series of power operators. In order to develop our formula, we written it in terms of the number operator and we used it to check the explicit expressions given in (Chair, 2013), and the well-known consequence says that the power of the bosonic operators is not bosonic operators is deduced.

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